Chapter 19

Matrices

19.1 $$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix} = (a_{ij})_{m\times n}$$

Notation for a *matrix*, where a_{ij} is the element in the ith row and the jth column. The matrix has *order* $m \times n$. If $m = n$, the matrix is *square* of order n.

19.2 $$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ 0 & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & a_{nn} \end{pmatrix}$$

An *upper triangular* matrix. (All elements below the diagonal are 0.) The transpose of $\mathbf{A}$ (see (19.11)) is called *lower triangular*.

19.3 $$\operatorname{diag}(a_1, a_2, \dots, a_n) = \begin{pmatrix} a_1 & 0 & \cdots & 0 \\ 0 & a_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_n \end{pmatrix}$$

A *diagonal matrix*.

19.4 $$\begin{pmatrix} a & 0 & \cdots & 0 \\ 0 & a & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a \end{pmatrix}_{n\times n}$$

A *scalar* matrix.

19.5 $$\mathbf{I}_n = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}_{n\times n}$$

The *unit* or *identity* matrix.

If $\mathbf{A} = (a_{ij})_{m\times n}$, $\mathbf{B} = (b_{ij})_{m\times n}$, and α is a scalar, we define

19.6 $$\mathbf{A} + \mathbf{B} = (a_{ij} + b_{ij})_{m\times n}$$
$$\alpha\mathbf{A} = (\alpha a_{ij})_{m\times n}$$
$$\mathbf{A} - \mathbf{B} = \mathbf{A} + (-1)\mathbf{B} = (a_{ij} - b_{ij})_{m\times n}$$

Matrix operations. (The scalars are real or complex numbers.)

19.7
$$(\mathbf{A}+\mathbf{B})+\mathbf{C}=\mathbf{A}+(\mathbf{B}+\mathbf{C})$$
$$\mathbf{A}+\mathbf{B}=\mathbf{B}+\mathbf{A}$$
$$\mathbf{A}+\mathbf{0}=\mathbf{A}$$
$$\mathbf{A}+(-\mathbf{A})=\mathbf{0}$$
$$(a+b)\mathbf{A}=a\mathbf{A}+b\mathbf{A}$$
$$a(\mathbf{A}+\mathbf{B})=a\mathbf{A}+a\mathbf{B}$$

Properties of matrix operations. $\mathbf{0}$ is the zero (or null) matrix, all of whose elements are zero. a and b are scalars.

19.8 If $\mathbf{A}=(a_{ij})_{m\times n}$ and $\mathbf{B}=(b_{jk})_{n\times p}$, we define the *product* $\mathbf{C}=\mathbf{AB}$ as the $m\times p$ matrix $\mathbf{C}=(c_{ik})_{m\times p}$ where

$$c_{ik}=a_{i1}b_{1k}+\cdots+a_{ij}b_{jk}+\cdots+a_{in}b_{nk}$$

The definition of *matrix multiplication*.

$$\begin{pmatrix} a_{11} & \dots & a_{1j} & \dots & a_{1n} \\ \vdots & & \vdots & & \vdots \\ \boxed{a_{i1}} & \boxed{\dots} & \boxed{a_{ij}} & \boxed{\dots} & \boxed{a_{in}} \\ \vdots & & \vdots & & \vdots \\ a_{m1} & \dots & a_{mj} & \dots & a_{mn} \end{pmatrix} \cdot \begin{pmatrix} b_{11} & \dots & \boxed{b_{1k}} & \dots & b_{1p} \\ \vdots & & \boxed{\vdots} & & \vdots \\ b_{j1} & \dots & \boxed{b_{jk}} & \dots & b_{jp} \\ \vdots & & \boxed{\vdots} & & \vdots \\ b_{n1} & \dots & \boxed{b_{nk}} & \dots & b_{np} \end{pmatrix} = \begin{pmatrix} c_{11} & \dots & c_{1k} & \dots & c_{1p} \\ \vdots & & \vdots & & \vdots \\ c_{i1} & \dots & \boxed{c_{ik}} & \dots & c_{ip} \\ \vdots & & \vdots & & \vdots \\ c_{m1} & \dots & c_{mk} & \dots & c_{mp} \end{pmatrix}$$

19.9
$$(\mathbf{AB})\mathbf{C}=\mathbf{A}(\mathbf{BC})$$
$$\mathbf{A}(\mathbf{B}+\mathbf{C})=\mathbf{AB}+\mathbf{AC}$$
$$(\mathbf{A}+\mathbf{B})\mathbf{C}=\mathbf{AC}+\mathbf{BC}$$

Properties of matrix multiplication.

19.10
$$\mathbf{AB}\neq\mathbf{BA}$$
$$\mathbf{AB}=\mathbf{0} \not\Rightarrow \mathbf{A}=\mathbf{0} \text{ or } \mathbf{B}=\mathbf{0}$$
$$\mathbf{AB}=\mathbf{AC} \;\&\; \mathbf{A}\neq\mathbf{0} \not\Rightarrow \mathbf{B}=\mathbf{C}$$

Important observations about matrix multiplication. $\mathbf{0}$ is the zero matrix. $\not\Rightarrow$ should be read: "does not necessarily imply".

19.11
$$\mathbf{A}'=\begin{pmatrix} a_{11} & a_{21} & \cdots & a_{m1} \\ a_{12} & a_{22} & \cdots & a_{m2} \\ \vdots & \vdots & & \vdots \\ a_{1n} & a_{2n} & \cdots & a_{mn} \end{pmatrix}$$

$\mathbf{A}'$, the *transpose* of $\mathbf{A}=(a_{ij})_{m\times n}$, is the $n\times m$ matrix obtained by interchanging rows and columns in $\mathbf{A}$.

19.12
$$(\mathbf{A}')'=\mathbf{A}$$
$$(\mathbf{A}+\mathbf{B})'=\mathbf{A}'+\mathbf{B}'$$
$$(\alpha\mathbf{A})'=\alpha\mathbf{A}'$$
$$(\mathbf{AB})'=\mathbf{B}'\mathbf{A}' \quad \text{(NOTE the order!)}$$

Rules for transposes.

19.13
$$\mathbf{B}=\mathbf{A}^{-1} \iff \mathbf{AB}=\mathbf{I}_n \iff \mathbf{BA}=\mathbf{I}_n$$

The *inverse* of an $n\times n$ matrix $\mathbf{A}$. $\mathbf{I}_n$ is the identity matrix.