Chapter 2

Equations. Functions of one variable. Complex numbers

2.1	$ax^2 + bx + c = 0 \iff x_{1,2} = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$	The roots of the general *quadratic* equation. They are real provided $b^2 \geq 4ac$ (assuming that a, b, and c are real).
2.2	If x_1 and x_2 are the roots of $x^2 + px + q = 0$, then $x_1 + x_2 = -p, \qquad x_1 x_2 = q$	Viète's rule.
2.3	$ax^3 + bx^2 + cx + d = 0$	The general *cubic* equation.
2.4	$x^3 + px + q = 0$	(2.3) reduces to the form (2.4) if x in (2.3) is replaced by $x - b/3a$.
2.5	$x^3 + px + q = 0$ with $\Delta = 4p^3 + 27q^2$ has • three different real roots if $\Delta < 0$; • three real roots, at least two of which are equal, if $\Delta = 0$; • one real and two complex roots if $\Delta > 0$.	Classification of the roots of (2.4) (assuming that p and q are real).
2.6	The solutions of $x^3 + px + q = 0$ are $x_1 = u + v$, $x_2 = \omega u + \omega^2 v$, and $x_3 = \omega^2 u + \omega v$, where $\omega = -\frac{1}{2} + \frac{i}{2}\sqrt{3}$, and $u = \sqrt[3]{-\dfrac{q}{2} + \dfrac{1}{2}\sqrt{\dfrac{4p^3 + 27q^2}{27}}}$ $v = \sqrt[3]{-\dfrac{q}{2} - \dfrac{1}{2}\sqrt{\dfrac{4p^3 + 27q^2}{27}}}$	*Cardano's formulas* for the roots of a cubic equation. i is the imaginary unit (see (2.75)) and ω is a complex third root of 1 (see (2.88)). (If complex numbers become involved, the cube roots must be chosen so that $3uv = -p$. Don't try to use these formulas unless you have to!)

2.7 If x_1, x_2, and x_3 are the roots of the equation $x^3 + px^2 + qx + r = 0$, then

$$x_1 + x_2 + x_3 = -p$$
$$x_1x_2 + x_1x_3 + x_2x_3 = q$$
$$x_1x_2x_3 = -r$$

Useful relations.

2.8 $P(x) = a_nx^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0$

A *polynomial* of degree n. ($a_n \neq 0$.)

2.9 For the polynomial $P(x)$ in (2.8) there exist constants $x_1, x_2, \ldots, x_n$ (real or complex) such that

$$P(x) = a_n(x - x_1)\cdots(x - x_n)$$

The *fundamental theorem of algebra.* $x_1, \ldots, x_n$ are called *zeros* of $P(x)$ and *roots* of $P(x) = 0$.

2.10
$$x_1 + x_2 + \cdots + x_n = -\frac{a_{n-1}}{a_n}$$
$$x_1x_2 + x_1x_3 + \cdots + x_{n-1}x_n = \sum_{i<j} x_ix_j = \frac{a_{n-2}}{a_n}$$
$$x_1x_2\cdots x_n = (-1)^n\frac{a_0}{a_n}$$

Relations between the roots and the coefficients of $P(x) = 0$, where $P(x)$ is defined in (2.8). (Generalizes (2.2) and (2.7).)

2.11 If $a_{n-1}, \ldots, a_1, a_0$ are all integers, then any integer root of the equation

$$x^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0 = 0$$

must divide a_0.

Any integer solutions of $x^3 + 6x^2 - x - 6 = 0$ must divide -6. (In this case the roots are ± 1 and -6.)

2.12 Let k be the number of changes of sign in the sequence of coefficients $a_n, a_{n-1}, \ldots, a_1, a_0$ in (2.8). The number of positive real roots of $P(x) = 0$, counting the multiplicities of the roots, is k or k minus a positive even number. If $k = 1$, the equation has exactly one positive real root.

Descartes's rule of signs.

2.13 The graph of the equation

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$$

is

- an ellipse, a point or empty if $4AC > B^2$;
- a parabola, a line, two parallel lines, or empty if $4AC = B^2$;
- a hyperbola or two intersecting lines if $4AC < B^2$.

Classification of *conics*. A, B, C not all 0.