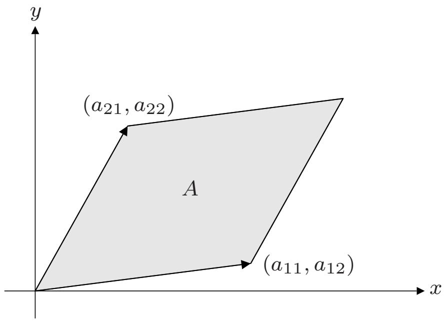
Chapter 20

Determinants

20.1 $$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{21}a_{12}$$

Definition of a 2×2 determinant.

20.2

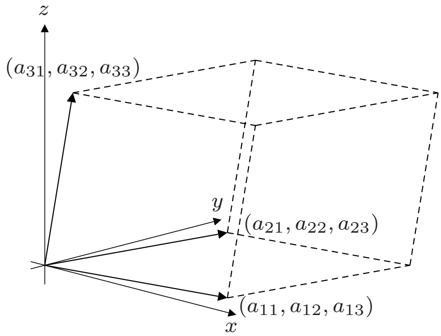


Geometric interpretation of a 2×2 determinant. The area A is the absolute value of the determinant
$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}.$$

20.3 $$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \begin{cases} \, a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32} \\ + a_{12}a_{23}a_{31} - a_{12}a_{21}a_{33} \\ + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31} \end{cases}$$

Definition of a 3×3 determinant.

20.4



Geometric interpretation of a 3×3 determinant. The volume of the "box" spanned by the three vectors is the absolute value of the determinant
$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}.$$

20.5 If $\mathbf{A} = (a_{ij})_{n\times n}$ is an $n \times n$ matrix, the *determinant* of $\mathbf{A}$ is the number

$$|\mathbf{A}| = a_{i1}A_{i1} + \cdots + a_{in}A_{in} = \sum_{j=1}^{n} a_{ij}A_{ij}$$

where A_{ij}, the cofactor of the element a_{ij}, is

$$A_{ij} = (-1)^{i+j} \begin{vmatrix} a_{11} & \dots & a_{1j} & \dots & a_{1n} \\ \vdots & & \vdots & & \vdots \\ a_{i1} & \dots & \boxed{a_{ij}} & \dots & a_{in} \\ \vdots & & \vdots & & \vdots \\ a_{n1} & \dots & a_{nj} & \dots & a_{nn} \end{vmatrix}$$

The general definition of a determinant of order n, by *cofactor expansion* along the ith row. The value of the determinant is independent of the choice of i.

20.6
$$\begin{aligned} a_{i1}A_{i1} + a_{i2}A_{i2} + \cdots + a_{in}A_{in} &= |\mathbf{A}| \\ a_{i1}A_{k1} + a_{i2}A_{k2} + \cdots + a_{in}A_{kn} &= 0 \quad \text{if } k \neq i \end{aligned}$$

$$\begin{aligned} a_{1j}A_{1j} + a_{2j}A_{2j} + \cdots + a_{nj}A_{nj} &= |\mathbf{A}| \\ a_{1j}A_{1k} + a_{2j}A_{2k} + \cdots + a_{nj}A_{nk} &= 0 \quad \text{if } k \neq j \end{aligned}$$

Expanding a determinant by a row or a column in terms of the cofactors of the same row or column, yields the determinant. Expanding by a row or a column in terms of the cofactors of a different row or column, yields 0.

20.7
- If all the elements in a row (or column) of $\mathbf{A}$ are 0, then $|\mathbf{A}| = 0$.
- If two rows (or two columns) of $\mathbf{A}$ are interchanged, the determinant changes sign but the absolute value remains unchanged.
- If all the elements in a single row (or column) of $\mathbf{A}$ are multiplied by a number c, the determinant is multiplied by c.
- If two of the rows (or columns) of $\mathbf{A}$ are proportional, then $|\mathbf{A}| = 0$.
- The value of $|\mathbf{A}|$ remains unchanged if a multiple of one row (or one column) is added to another row (or column).
- $|\mathbf{A}'| = |\mathbf{A}|$, where $\mathbf{A}'$ is the transpose of $\mathbf{A}$.

Important properties of determinants. $\mathbf{A}$ is a square matrix.

20.8
$$|\mathbf{AB}| = |\mathbf{A}| \cdot |\mathbf{B}|$$
$$|\mathbf{A} + \mathbf{B}| \neq |\mathbf{A}| + |\mathbf{B}| \quad \text{(in general)}$$

Properties of determinants. $\mathbf{A}$ and $\mathbf{B}$ are $n \times n$ matrices.

20.9
$$\begin{vmatrix} 1 & x_1 & x_1^2 \\ 1 & x_2 & x_2^2 \\ 1 & x_3 & x_3^2 \end{vmatrix} = (x_2 - x_1)(x_3 - x_1)(x_3 - x_2)$$

The *Vandermonde determinant* for $n = 3$.