Chapter 21

Eigenvalues. Quadratic forms

21.1	A scalar λ is called an *eigenvalue* of an $n \times n$ matrix $\mathbf{A}$ if there exists an n-vector $\mathbf{c} \neq \mathbf{0}$ such that $\mathbf{A}\mathbf{c} = \lambda\mathbf{c}$. The vector $\mathbf{c}$ is called an *eigenvector* of $\mathbf{A}$.	Eigenvalues and eigenvectors are also called *characteristic roots* and *characteristic vectors*. λ and $\mathbf{c}$ may be complex even if $\mathbf{A}$ is real.
21.2	$\lvert\mathbf{A} - \lambda\mathbf{I}\rvert = \begin{vmatrix} a_{11}-\lambda & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22}-\lambda & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn}-\lambda \end{vmatrix}$	The *eigenvalue polynomial* (the *characteristic polynomial*) of $\mathbf{A} = (a_{ij})_{n\times n}$. $\mathbf{I}$ is the unit matrix of order n.
21.3	λ is an eigenvalue of $\mathbf{A} \Leftrightarrow\ p(\lambda) = \lvert\mathbf{A} - \lambda\mathbf{I}\rvert = 0$	A necessary and sufficient condition for λ to be an eigenvalue of $\mathbf{A}$.
21.4	$\lvert\mathbf{A}\rvert = \lambda_1 \cdot \lambda_2 \cdots \lambda_{n-1} \cdot \lambda_n$ $\mathrm{tr}(\mathbf{A}) = a_{11} + \cdots + a_{nn} = \lambda_1 + \cdots + \lambda_n$	$\lambda_1, \dots, \lambda_n$ are the eigenvalues of $\mathbf{A}$.
21.5	Let $f(\ )$ be a polynomial. If λ is an eigenvalue of $\mathbf{A}$, then $f(\lambda)$ is an eigenvalue of $f(\mathbf{A})$.	Eigenvalues for matrix polynomials.
21.6	A square matrix $\mathbf{A}$ has an inverse if and only if 0 is not an eigenvalue of $\mathbf{A}$. If $\mathbf{A}$ has an inverse and λ is an eigenvalue of $\mathbf{A}$, then λ^{-1} is an eigenvalue of $\mathbf{A}^{-1}$.	How to find the eigenvalues of the inverse of a square matrix.
21.7	All eigenvalues of $\mathbf{A}$ have moduli (strictly) less than 1 if and only if $\mathbf{A}^t \to \mathbf{0}$ as $t \to \infty$.	An important result.
21.8	$\mathbf{AB}$ and $\mathbf{BA}$ have the same eigenvalues.	$\mathbf{A}$ and $\mathbf{B}$ are $n \times n$ matrices.
21.9	If $\mathbf{A}$ is symmetric and has only real elements, then all eigenvalues of $\mathbf{A}$ are reals.	

21.10 If

$p(\lambda) = |\mathbf{A} - \lambda \mathbf{I}| =$

$$(-\lambda)^n + b_{n-1}(-\lambda)^{n-1} + \cdots + b_1(-\lambda) + b_0$$

is the eigenvalue polynomial of $\mathbf{A}$, then b_k is the sum of all principal minors of $\mathbf{A}$ of order $n-k$ (there are $\binom{n}{k}$ of them).

Characterization of the coefficients of the eigenvalue polynomial of an $n \times n$ matrix $\mathbf{A}$. (For principal minors, see (20.15).) $p(\lambda) = 0$ is called the *eigenvalue equation* or *characteristic equation* of $\mathbf{A}$.

21.11
$$\begin{vmatrix} a_{11}-\lambda & a_{12} \\ a_{21} & a_{22}-\lambda \end{vmatrix} = (-\lambda)^2 + b_1(-\lambda) + b_0$$

where $b_1 = a_{11} + a_{22} = \mathrm{tr}(\mathbf{A})$, $b_0 = |\mathbf{A}|$

(21.10) for $n = 2$. ($\mathrm{tr}(\mathbf{A})$ is the trace of $\mathbf{A}$.)

21.12
$$\begin{vmatrix} a_{11}-\lambda & a_{12} & a_{13} \\ a_{21} & a_{22}-\lambda & a_{23} \\ a_{31} & a_{32} & a_{33}-\lambda \end{vmatrix} =$$

$$(-\lambda)^3 + b_2(-\lambda)^2 + b_1(-\lambda) + b_0$$

where

$$b_2 = a_{11} + a_{22} + a_{33} = \mathrm{tr}(\mathbf{A})$$

$$b_1 = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} + \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} + \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$$

$$b_0 = |\mathbf{A}|$$

(21.10) for $n = 3$.

21.13 $\mathbf{A}$ is *diagonalizable* $\Leftrightarrow$ $\left\{\begin{array}{l} \mathbf{P}^{-1}\mathbf{A}\mathbf{P} = \mathbf{D} \text{ for some matrix } \mathbf{P} \text{ and some diagonal matrix } \mathbf{D}. \end{array}\right.$

A definition.

21.14 $\mathbf{A}$ and $\mathbf{P}^{-1}\mathbf{A}\mathbf{P}$ have the same eigenvalues.

21.15 If $\mathbf{A} = (a_{ij})_{n \times n}$ has n distinct eigenvalues, then $\mathbf{A}$ is diagonalizable.

Sufficient (but NOT necessary) condition for $\mathbf{A}$ to be diagonalizable.

21.16 $\mathbf{A} = (a_{ij})_{n \times n}$ has n linearly independent eigenvectors, $\mathbf{x}_1, \ldots, \mathbf{x}_n$, with corresponding eigenvalues $\lambda_1, \ldots, \lambda_n$, if and only if

$$\mathbf{P}^{-1}\mathbf{A}\mathbf{P} = \begin{pmatrix} \lambda_1 & 0 & \ldots & 0 \\ 0 & \lambda_2 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & \lambda_n \end{pmatrix}$$

where $\mathbf{P} = (\mathbf{x}_1, \ldots, \mathbf{x}_n)_{n \times n}$.

A characterization of diagonalizable matrices.