Chapter 22

Special matrices. Leontief systems

Idempotent matrices

22.1	$\mathbf{A} = (a_{ij})_{n\times n}$ is *idempotent* $\iff$ $\mathbf{A}^2 = \mathbf{A}$	Definition of an idempotent matrix.
22.2	$\mathbf{A}$ is idempotent $\iff$ $\mathbf{I} - \mathbf{A}$ is idempotent.	Properties of idempotent matrices.
22.3	$\mathbf{A}$ is idempotent $\Rightarrow$ 0 and 1 are the only possible eigenvalues, and $\mathbf{A}$ is positive semidefinite.	
22.4	$\mathbf{A}$ is idempotent with k eigenvalues equal to 1 $\Rightarrow$ $r(\mathbf{A}) = \mathrm{tr}(\mathbf{A}) = k$.	
22.5	$\mathbf{A}$ is idempotent and $\mathbf{C}$ is orthogonal $\Rightarrow$ $\mathbf{C}'\mathbf{A}\mathbf{C}$ is idempotent.	An orthogonal matrix is defined in (22.8).
22.6	$\mathbf{A}$ is idempotent $\iff$ its associated linear transformation is a projection.	A linear transformation P from $\mathbb{R}^n$ into $\mathbb{R}^n$ is a *projection* if $P(P(\mathbf{x})) = P(\mathbf{x})$ for all $\mathbf{x}$ in $\mathbb{R}^n$.
22.7	$\mathbf{I}_n - \mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'$ is idempotent.	$\mathbf{X}$ is $n \times m$, $\lvert\mathbf{X}'\mathbf{X}\rvert \neq 0$.

Orthogonal matrices

22.8	$\mathbf{P} = (p_{ij})_{n\times n}$ is *orthogonal* $\iff \mathbf{P}'\mathbf{P} = \mathbf{P}\mathbf{P}' = \mathbf{I}_n$	Definition of an orthogonal matrix.
22.9	$\mathbf{P}$ is orthogonal $\iff$ the column vectors of $\mathbf{P}$ are mutually orthogonal unit vectors.	A property of orthogonal matrices.

22.10 $\mathbf{P}$ and $\mathbf{Q}$ are orthogonal $\Rightarrow$ $\mathbf{PQ}$ is orthogonal.

Properties of orthogonal matrices.

22.11 $\mathbf{P}$ orthogonal $\Rightarrow |\mathbf{P}| = \pm 1$, and 1 and -1 are the only possible real eigenvalues.

22.12 $\mathbf{P}$ orthogonal $\Leftrightarrow$ $\|\mathbf{Px}\| = \|\mathbf{x}\|$ for all $\mathbf{x}$ in $\mathbb{R}^n$.

Orthogonal transformations preserve lengths of vectors.

22.13 If $\mathbf{P}$ is orthogonal, the angle between $\mathbf{Px}$ and $\mathbf{Py}$ equals the angle between $\mathbf{x}$ and $\mathbf{y}$.

Orthogonal transformations preserve angles.

Permutation matrices

22.14 $\mathbf{P} = (p_{ij})_{n\times n}$ is a *permutation* matrix if in each row and each column of $\mathbf{P}$ there is one element equal to 1 and the rest of the elements are 0.

Definition of a permutation matrix.

22.15 $\mathbf{P}$ is a permutation matrix $\Rightarrow$ $\mathbf{P}$ is nonsingular and orthogonal.

Properties of permutation matrices.

Nonnegative matrices

22.16
$$\mathbf{A} = (a_{ij})_{m\times n} \geq \mathbf{0} \iff a_{ij} \geq 0 \text{ for all } i, j$$
$$\mathbf{A} = (a_{ij})_{m\times n} > \mathbf{0} \iff a_{ij} > 0 \text{ for all } i, j$$

Definitions of *nonnegative* and *positive* matrices.

22.17 If $\mathbf{A} = (a_{ij})_{n\times n} \geq 0$, $\mathbf{A}$ has at least one nonnegative eigenvalue. The largest nonnegative eigenvalue is called the *Frobenius root* of $\mathbf{A}$ and it is denoted by $\lambda(\mathbf{A})$. $\mathbf{A}$ has a nonnegative eigenvector corresponding to $\lambda(\mathbf{A})$.

Definition of the Frobenius root (or *dominant root*) of a nonnegative matrix.

22.18
- μ is an eigenvalue of $\mathbf{A}$ $\Rightarrow$ $|\mu| \leq \lambda(\mathbf{A})$
- $\mathbf{0} \leq \mathbf{A}_1 \leq \mathbf{A}_2$ $\Rightarrow$ $\lambda(\mathbf{A}_1) \leq \lambda(\mathbf{A}_2)$
- $\rho > \lambda(\mathbf{A})$ $\Leftrightarrow$ $(\rho\mathbf{I} - \mathbf{A})^{-1}$ exists and is $\geq \mathbf{0}$
- $\displaystyle \min_{1\leq j\leq n} \sum_{i=1}^{n} a_{ij} \leq \lambda(\mathbf{A}) \leq \max_{1\leq j\leq n} \sum_{i=1}^{n} a_{ij}$

Properties of nonnegative matrices. $\lambda(\mathbf{A})$ is the Frobenius root of A.