Chapter 23

Kronecker products and the vec operator. Differentiation of vectors and matrices

23.1	$\mathbf{A} \otimes \mathbf{B} = \begin{pmatrix} a_{11}\mathbf{B} & a_{12}\mathbf{B} & \dots & a_{1n}\mathbf{B} \\ a_{21}\mathbf{B} & a_{22}\mathbf{B} & \dots & a_{2n}\mathbf{B} \\ \vdots & \vdots & & \vdots \\ a_{m1}\mathbf{B} & a_{m2}\mathbf{B} & \dots & a_{mn}\mathbf{B} \end{pmatrix}$	The *Kronecker product* of $\mathbf{A} = (a_{ij})_{m \times n}$ and $\mathbf{B} = (b_{ij})_{p \times q}$. $\mathbf{A} \otimes \mathbf{B}$ is $mp \times nq$. In general, the Kronecker product is not commutative, $\mathbf{A} \otimes \mathbf{B} \neq \mathbf{B} \otimes \mathbf{A}$.
23.2	$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \otimes \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} = \begin{pmatrix} a_{11}b_{11} & a_{11}b_{12} & a_{12}b_{11} & a_{12}b_{12} \\ a_{11}b_{21} & a_{11}b_{22} & a_{12}b_{21} & a_{12}b_{22} \\ a_{21}b_{11} & a_{21}b_{12} & a_{22}b_{11} & a_{22}b_{12} \\ a_{21}b_{21} & a_{21}b_{22} & a_{22}b_{21} & a_{22}b_{22} \end{pmatrix}$	A special case of (23.1).
23.3	$\mathbf{A} \otimes \mathbf{B} \otimes \mathbf{C} = (\mathbf{A} \otimes \mathbf{B}) \otimes \mathbf{C} = \mathbf{A} \otimes (\mathbf{B} \otimes \mathbf{C})$	Valid in general.
23.4	$(\mathbf{A} + \mathbf{B}) \otimes (\mathbf{C} + \mathbf{D}) = \mathbf{A} \otimes \mathbf{C} + \mathbf{A} \otimes \mathbf{D} + \mathbf{B} \otimes \mathbf{C} + \mathbf{B} \otimes \mathbf{D}$	Valid if $\mathbf{A}+\mathbf{B}$ and $\mathbf{C}+\mathbf{D}$ are defined.
23.5	$(\mathbf{A} \otimes \mathbf{B})(\mathbf{C} \otimes \mathbf{D}) = \mathbf{AC} \otimes \mathbf{BD}$	Valid if $\mathbf{AC}$ and $\mathbf{BD}$ are defined.
23.6	$(\mathbf{A} \otimes \mathbf{B})' = \mathbf{A}' \otimes \mathbf{B}'$	Rule for transposing a Kronecker product.
23.7	$(\mathbf{A} \otimes \mathbf{B})^{-1} = \mathbf{A}^{-1} \otimes \mathbf{B}^{-1}$	Valid if $\mathbf{A}^{-1}$ and $\mathbf{B}^{-1}$ exist.
23.8	$\operatorname{tr}(\mathbf{A} \otimes \mathbf{B}) = \operatorname{tr}(\mathbf{A}) \operatorname{tr}(\mathbf{B})$	$\mathbf{A}$ and $\mathbf{B}$ are square matrices, not necessarily of the same order.

23.9	$\alpha \otimes \mathbf{A} = \alpha\mathbf{A} = \mathbf{A}\alpha = \mathbf{A} \otimes \alpha$	α is a 1×1 scalar matrix.
23.10	If $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of $\mathbf{A}$, and if $\mu_1, \ldots, \mu_p$ are the eigenvalues of $\mathbf{B}$, then the np eigenvalues of $\mathbf{A} \otimes \mathbf{B}$ are $\lambda_i \mu_j$, $i = 1, \ldots, n$, $j = 1, \ldots, p$.	The eigenvalues of $\mathbf{A} \otimes \mathbf{B}$, where $\mathbf{A}$ is $n \times n$ and $\mathbf{B}$ is $p \times p$.
23.11	If $\mathbf{x}$ is an eigenvector of $\mathbf{A}$, and $\mathbf{y}$ is an eigenvector for $\mathbf{B}$, then $\mathbf{x} \otimes \mathbf{y}$ is an eigenvector of $\mathbf{A} \otimes \mathbf{B}$.	NOTE: An eigenvector of $\mathbf{A} \otimes \mathbf{B}$ is not necessarily the Kronecker product of an eigenvector of $\mathbf{A}$ and an eigenvector of $\mathbf{B}$.
23.12	If $\mathbf{A}$ and $\mathbf{B}$ are positive (semi-)definite, then $\mathbf{A} \otimes \mathbf{B}$ is positive (semi-)definite.	Follows from (23.10).
23.13	$\lvert\mathbf{A} \otimes \mathbf{B}\rvert = \lvert\mathbf{A}\rvert^p \cdot \lvert\mathbf{B}\rvert^n$	$\mathbf{A}$ is $n \times n$, $\mathbf{B}$ is $p \times p$.
23.14	$r(\mathbf{A} \otimes \mathbf{B}) = r(\mathbf{A})\, r(\mathbf{B})$	The rank of a Kronecker product.
23.15	If $\mathbf{A} = (\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n)_{m\times n}$, then $\operatorname{vec}(\mathbf{A}) = \begin{pmatrix} \mathbf{a}_1 \\ \mathbf{a}_2 \\ \vdots \\ \mathbf{a}_n \end{pmatrix}_{mn\times 1}$	$\operatorname{vec}(\mathbf{A})$ consists of the columns of $\mathbf{A}$ placed below each other.
23.16	$\operatorname{vec}\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{pmatrix} a_{11} \\ a_{21} \\ a_{12} \\ a_{22} \end{pmatrix}$	A special case of (23.15).
23.17	$\operatorname{vec}(\mathbf{A} + \mathbf{B}) = \operatorname{vec}(\mathbf{A}) + \operatorname{vec}(\mathbf{B})$	Valid if $\mathbf{A} + \mathbf{B}$ is defined.
23.18	$\operatorname{vec}(\mathbf{ABC}) = (\mathbf{C}' \otimes \mathbf{A}) \operatorname{vec}(\mathbf{B})$	Valid if the product $\mathbf{ABC}$ is defined.
23.19	$\operatorname{tr}(\mathbf{AB}) = (\operatorname{vec}(\mathbf{A}'))' \operatorname{vec}(\mathbf{B}) = (\operatorname{vec}(\mathbf{B}'))' \operatorname{vec}(\mathbf{A})$	Valid if the operations are defined.