Chapter 3

Limits. Continuity. Differentiation (one variable)

3.1 $f(x)$ tends to A as a *limit* as x approaches a,

$$\lim_{x\to a} f(x) = A \text{ or } f(x) \to A \text{ as } x \to a$$

if for every number $\varepsilon > 0$ there exists a number $\delta > 0$ such that

$$|f(x) - A| < \varepsilon \text{ if } x \in D_f \text{ and } 0 < |x - a| < \delta$$

The definition of a limit of a function of one variable. D_f is the domain of f.

3.2 If $\lim_{x\to a} f(x) = A$ and $\lim_{x\to a} g(x) = B$, then

- $\lim\limits_{x\to a} \big(f(x) \pm g(x)\big) = A \pm B$
- $\lim\limits_{x\to a} \big(f(x) \cdot g(x)\big) = A \cdot B$
- $\lim\limits_{x\to a} \dfrac{f(x)}{g(x)} = \dfrac{A}{B} \qquad (\text{if } B \neq 0)$

Rules for limits.

3.3 f is *continuous* at $x = a$ if $\lim\limits_{x\to a} f(x) = f(a)$, i.e. if $a \in D_f$ and for each number $\varepsilon > 0$ there is a number $\delta > 0$ such that

$$|f(x) - A| < \varepsilon \text{ if } x \in D_f \text{ and } |x - a| < \delta$$

f is *continuous on a set* $S \subset D_f$ if f is continuous at each point of S.

Definition of continuity.

3.4 If f and g are continuous at a, then:

- $f \pm g$ and $f \cdot g$ are continuous at a.
- f/g is continuous at a if $g(a) \neq 0$.

Properties of continuous functions.

3.5 If g is continuous at a, and f is continuous at $g(a)$, then $f(g(x))$ is continuous at a.

Continuity of *composite* functions.

3.6 Any function built from continuous functions by additions, subtractions, multiplications, divisions, and compositions, is continuous where defined.

A useful result.

3.7 f is *uniformly continuous* on a set S if for each $\varepsilon > 0$ there exists a $\delta > 0$ (depending on ε but NOT on x and y) such that

$$|f(x) - f(y)| < \varepsilon \text{ if } x,\, y \in S \text{ and } |x - y| < \delta$$

Definition of uniform continuity.

3.8 If f is continuous on a closed bounded interval I, then f is uniformly continuous on I.

Continuous functions on closed bounded intervals are uniformly continuous.

3.9 If f is continuous on an interval I containing a and b, and A lies between $f(a)$ and $f(b)$, then there is at least one ξ between a and b such that $A = f(\xi)$.

The *intermediate value theorem.*

3.10

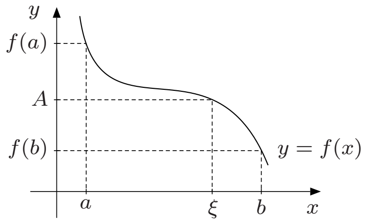

Illustration of the intermediate value theorem.

3.11 $$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

The definition of the *derivative*. If the limit exists, f is called *differentiable* at x.

3.12 Other notations for the derivative of $y = f(x)$ include

$$f'(x) = y' = \frac{dy}{dx} = \frac{df(x)}{dx} = Df(x)$$

Other notations for the derivative.

3.13 $y = f(x) \pm g(x) \;\Rightarrow\; y' = f'(x) \pm g'(x)$

General rules.

3.14 $y = f(x)g(x) \;\Rightarrow\; y' = f'(x)g(x) + f(x)g'(x)$

3.15 $$y = \frac{f(x)}{g(x)} \;\Rightarrow\; y' = \frac{f'(x)g(x) - f(x)g'(x)}{\big(g(x)\big)^2}$$

3.16 $y = f\big(g(x)\big) \;\Rightarrow\; y' = f'(g(x)) \cdot g'(x)$

The *chain rule.*

3.17 $y = f(x)^{g(x)} \;\Rightarrow$

$$y' = f(x)^{g(x)} \left(g'(x) \ln f(x) + g(x) \frac{f'(x)}{f(x)} \right)$$

A useful formula.