Chapter 31

Non-cooperative game theory

31.1 In an n-person game we assign to each player i ($i = 1, \ldots, n$) a *strategy set* S_i and a pure strategy payoff function u_i that gives player i utility $u_i(\mathbf{s}) = u_i(s_1, \ldots, s_n)$ for each *strategy profile* $\mathbf{s} = (s_1, \ldots, s_n) \in S = S_1 \times \cdots \times S_n$.

An *n-person game* in *strategic* (or *normal*) form. If all the strategy sets S_i have a finite number of elements, the game is called *finite*.

31.2 A strategy profile $(s_1^*, \ldots, s_n^*)$ for an n-person game is a *pure strategy Nash equilibrium* if for all $i = 1, \ldots, n$ and all s_i in S_i,

$$u_i(s_1^*, \ldots, s_n^*) \geq u_i(s_1^*, \ldots, s_{i-1}^*, s_i, s_{i+1}^*, \ldots s_n^*)$$

Definition of a pure strategy Nash equilibrium for an n-person game.

31.3 If for all $i = 1, \ldots, n$, the strategy set S_i is a nonempty, compact, and convex subset of $\mathbb{R}^m$, and $u_i(s_1, \ldots, s_n)$ is continuous in $S = S_1 \times \cdots \times S_n$ and quasiconcave in its ith variable, then the game has a pure strategy Nash equilibrium.

Sufficient conditions for the existence of a pure strategy Nash equilibrium. (There will usually be several Nash equilibria.)

31.4 Consider a finite n-person game where S_i is player i's pure strategy set, and let $S = S_1 \times \cdots \times S_n$. Let Ω_i be a set of probability distributions over S_i. An element σ_i of Ω_i (σ_i is then a function $\sigma_i : S_i \to [0, 1]$) is called a *mixed strategy* for player i, with the interpretation that if i plays σ_i, then i chooses the pure strategy s_i with probability $\sigma_i(s_i)$.

If the players choose the *mixed strategy profile* $\boldsymbol{\sigma} = (\sigma_1, \ldots, \sigma_n) \in \Omega_1 \times \cdots \times \Omega_n$, the probability that the pure strategy profile $\mathbf{s} = (s_1, \ldots, s_n)$ occurs is $\sigma_1(s_1) \cdots \sigma_n(s_n)$. The expected payoff to player i is then

$$u_i(\boldsymbol{\sigma}) = \sum_{s \in S} \sigma_1(s_1) \cdots \sigma_n(s_n) u_i(s)$$

Definition of a *mixed strategy* for an n-person game.

31.5 A mixed strategy profile $\boldsymbol{\sigma}^* = (\sigma_1^*, \dots, \sigma_n^*)$ is a *Nash equilibrium* if for all i and every σ_i,

$$u_i(\boldsymbol{\sigma}^*) \geq u_i(\sigma_1^*, \dots, \sigma_{i-1}^*, \sigma_i, \sigma_{i+1}^*, \dots, \sigma_n^*)$$

Definition of a *mixed strategy Nash equilibrium* for an n-person game.

31.6 $\boldsymbol{\sigma}^*$ is a Nash equilibrium if and only if the following conditions hold for all $i = 1, \dots, n$:

$$\sigma_i^*(s_i) > 0 \Rightarrow u_i(\boldsymbol{\sigma}^*) = u_i(s_i, \boldsymbol{\sigma}_{-i}^*) \quad \text{for all } s_i$$

$$\sigma_i^*(s_i') = 0 \Rightarrow u_i(\boldsymbol{\sigma}^*) \geq u_i(s_i', \boldsymbol{\sigma}_{-i}^*) \quad \text{for all } s_i'$$

where $\boldsymbol{\sigma}_{-i}^* = (\sigma_1^*, \dots, \sigma_{i-1}^*, \sigma_{i+1}^*, \dots, \sigma_n^*)$ and we consider s_i and s_i' as degenerate mixed strategies.

An equivalent definition of a (mixed strategy) Nash equilibrium.

31.7 Every finite n-person game has a mixed strategy Nash equilibrium.

An important result.

31.8 The pure strategy $s_i \in S_i$ of player i is *strictly dominated* if there exists a mixed strategy σ_i for player i such that for all feasible combinations of the other players' pure strategies, i's payoff from playing strategy s_i is strictly less than i's payoff from playing σ_i:

$$u_i(s_1, \dots, s_{i-1}, s_i, s_{i+1}, \dots, s_n) <$$
$$u_i(s_1, \dots, s_{i-1}, \sigma_i, s_{i+1}, \dots, s_n)$$

for every $(s_1, \dots, s_{i-1}, s_{i+1}, \dots, s_n)$ that can be constructed from the other players' strategy sets $S_1, \dots, S_{i-1}, S_{i+1}, \dots, S_n$.

Definition of strictly dominated strategies.

31.9 In an n-person game, the following results hold:

- If iterated elimination of strictly dominated strategies eliminates all but the strategies $(s_1^*, \dots, s_n^*)$, then these strategies are the unique Nash equilibrium of the game.
- If the mixed strategy profile $(\sigma_1^*, \dots, \sigma_n^*)$ is a Nash equilibrium and, for some player i, $\sigma_i^*(s_i) > 0$, then s_i survives iterated elimination of strictly dominated strategies.

Useful results. Iterated elimination of strictly dominated strategies need not result in the elimination of any strategy. (For a discussion of iterated elimination of strictly dominated strategies, see the literature.)