Chapter 33

Probability and statistics

33.1 The probability $P(A)$ of an event $A \subset \Omega$ satisfies the following axioms:

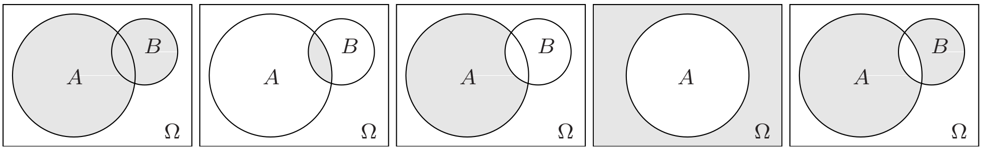
(a) $0 \leq P(A) \leq 1$

(b) $P(\Omega) = 1$

(c) If $A_i \cap A_j = \varnothing$ for $i \neq j$, then

$$P\Big(\bigcup_{i=1}^{\infty} A_i\Big) = \sum_{i=1}^{\infty} P(A_i)$$

Axioms for probability. Ω is the sample space consisting of all possible outcomes. An event is a subset of Ω.



$A \cup B$ — A or B occurs

$A \cap B$ — Both A and B occur

$A \setminus B$ — A occurs, but B does not

A^c — A does not occur

$A \triangle B$ — A or B occurs, but not both

33.2

- $P(A^c) = 1 - P(A)$
- $P(A \cup B) = P(A) + P(B) - P(A \cap B)$
- $P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(A \cap C) - P(B \cap C) + P(A \cap B \cap C)$
- $P(A \setminus B) = P(A) - P(A \cap B)$
- $P(A \triangle B) = P(A) + P(B) - 2P(A \cap B)$

Rules for calculating probabilities.

33.3 $P(A \mid B) = \dfrac{P(A \cap B)}{P(B)}$ is the *conditional probability* that event A will occur given that B has occurred.

Definition of *conditional probability*, $P(B) > 0$.

33.4 A and B are *(stochastically) independent* if

$$P(A \cap B) = P(A)P(B)$$

If $P(B) > 0$, this is equivalent to

$$P(A \mid B) = P(A)$$

Definition of (stochastic) independence.

33.5 $P(A_1 \cap A_2 \cap \ldots \cap A_n) = P(A_1)P(A_2 \,|\, A_1) \cdots P(A_n \,|\, A_1 \cap A_2 \cap \cdots \cap A_{n-1})$

General multiplication rule for probabilities.

33.6 $$P(A \,|\, B) = \frac{P(B \,|\, A) \cdot P(A)}{P(B)} = \frac{P(B \,|\, A)P(A)}{P(B \,|\, A)P(A) + P(B \,|\, A^c)P(A^c)}$$

Bayes's rule. ($P(B) \neq 0$.)

33.7 $$P(A_i \,|\, B) = \frac{P(B \,|\, A_i) \cdot P(A_i)}{\sum\limits_{j=1}^{n} P(B \,|\, A_j) \cdot P(A_j)}$$

Generalized *Bayes's rule*. $A_1, \ldots, A_n$ are disjoint, $\sum_{i=1}^{n} P(A_i) = P(\Omega) = 1$, where $\Omega = \bigcup_{i=1}^{n} A_i$ is the sample space. B is an arbitrary event.

One-dimensional random variables

33.8
- $P(X \in A) = \sum\limits_{x \in A} f(x)$
- $P(X \in A) = \int\limits_{A} f(x)\,dx$

f is the discrete/continuous probability density function for the random (or stochastic) variable X.

33.9
- $F(x) = P(X \leq x) = \sum\limits_{t \leq x} f(t)$
- $F(x) = P(X \leq x) = \int\limits_{-\infty}^{x} f(t)\,dt$

F is the cumulative discrete/continuous distribution function. In the continuous case, $P(X = x) = 0$.

33.10
- $E[X] = \sum\limits_{x} x f(x)$
- $E[X] = \int\limits_{-\infty}^{\infty} x f(x)\,dx$

Expectation of a random variable X with discrete/continuous probability density function f. $\mu = E[X]$ is called the *mean*.

33.11
- $E[g(X)] = \sum\limits_{x} g(x) f(x)$
- $E[g(X)] = \int\limits_{-\infty}^{\infty} g(x) f(x)\,dx$

Expectation of a function g of a random variable X with discrete/continuous probability density function f.

33.12 $\mathrm{var}[X] = E[(X - E[X])^2]$

The *variance* of a random variable is, by definition, the expected value of its squared deviation from the mean.