Chapter 34

Probability distributions

34.1 $$f(x) = \begin{cases} \dfrac{x^{p-1}(1-x)^{q-1}}{B(p,q)}, & x \in (0,1), \\ 0 & \text{otherwise,} \end{cases}$$

$p > 0$, $q > 0$.

Mean: $E[X] = \dfrac{p}{p+q}$.

Variance: $\operatorname{var}[X] = \dfrac{pq}{(p+q)^2(p+q+1)}$.

kth moment: $E[X^k] = \dfrac{B(p+k,q)}{B(p,q)}$.

Beta distribution. B is the beta function defined in (9.61).

34.2 $$f(x) = \binom{n}{x} p^x (1-p)^{n-x},$$

$x = 0, 1, \ldots, n;\quad n = 1, 2, \ldots;\quad p \in (0,1).$

Mean: $E[X] = np$.
Variance: $\operatorname{var}[X] = np(1-p)$.
Moment generating function: $[pe^t + (1-p)]^n$.
Characteristic function: $[pe^{it} + (1-p)]^n$.

Binomial distribution. $f(x)$ is the probability for an event to occur exactly x times in n independent observations, when the probability of the event is p at each observation. For $\binom{n}{x}$, see (8.30).

34.3 $$f(x,y) = \frac{e^{-Q}}{2\pi\sigma\tau\sqrt{1-\rho^2}}, \quad \text{where}$$

$$Q = \frac{\left(\frac{x-\mu}{\sigma}\right)^2 - 2\rho\frac{(x-\mu)(y-\eta)}{\sigma\tau} + \left(\frac{y-\eta}{\tau}\right)^2}{2(1-\rho^2)},$$

$x, y, \mu, \eta \in (-\infty, \infty),\ \sigma > 0,\ \tau > 0,\ |\rho| < 1.$

Mean: $E[X] = \mu$, $E[Y] = \eta$.
Variance: $\operatorname{var}[X] = \sigma^2$, $\operatorname{var}[Y] = \tau^2$.
Covariance: $\operatorname{cov}[X,Y] = \rho\sigma\tau$.

Binormal distribution. (For moment generating and characteristic functions, see the more general multivariate normal distribution in (34.15).)

34.4	$$f(x)=\begin{cases}\dfrac{x^{\frac{1}{2}\nu-1}e^{-\frac{1}{2}x}}{2^{\frac{1}{2}\nu}\Gamma(\frac{1}{2}\nu)}, & x>0\\ 0, & x\le 0\end{cases};\quad \nu=1,2,\ldots$$ Mean: $E[X]=\nu$. Variance: $\operatorname{var}[X]=2\nu$. Moment generating function: $(1-2t)^{-\frac{1}{2}\nu},\ t<\frac{1}{2}$. Characteristic function: $(1-2it)^{-\frac{1}{2}\nu}$.	*Chi-square distribution* with ν degrees of freedom. Γ is the gamma function defined in (9.53).
34.5	$$f(x)=\begin{cases}\lambda e^{-\lambda x}, & x>0\\ 0, & x\le 0\end{cases};\qquad \lambda>0.$$ Mean: $E[X]=1/\lambda$. Variance: $\operatorname{var}[X]=1/\lambda^2$. Moment generating function: $\lambda/(\lambda-t),\ t<\lambda$. Characteristic function: $\lambda/(\lambda-it)$.	*Exponential distribution.*
34.6	$$f(x)=\frac{1}{\beta}e^{-z}e^{-e^{-z}},\quad z=\frac{x-\alpha}{\beta},\quad x\in\mathbb{R},\quad \beta>0$$ Mean: $E[X]=\alpha-\beta\Gamma'(1)$. Variance: $\operatorname{var}[X]=\beta^2\pi^2/6$. Moment gen. function: $e^{\alpha t}\Gamma(1-\beta t),\ t<1/\beta$. Characteristic function: $e^{i\alpha t}\Gamma(1-i\beta t)$.	*Extreme value (Gumbel) distribution.* $\Gamma'(1)$ is the derivative of the gamma function at 1. (See (9.53).) $\Gamma'(1)=-\gamma$, where $\gamma\approx 0.5772$ is Euler's constant, see (8.48).
34.7	$$f(x)=\begin{cases}\dfrac{\nu_1^{\frac{1}{2}\nu_1}\nu_2^{\frac{1}{2}\nu_2}x^{\frac{1}{2}\nu_1-1}}{B(\frac{1}{2}\nu_1,\frac{1}{2}\nu_2)(\nu_2+\nu_1 x)^{\frac{1}{2}(\nu_1+\nu_2)}}, & x>0\\ 0, & x\le 0\end{cases}$$ $\nu_1,\nu_2=1,2,\ldots$ Mean: $E[X]=\nu_2/(\nu_2-2)$ for $\nu_2>2$ (does not exist for $\nu_2=1,2$). Variance: $\operatorname{var}[X]=\dfrac{2\nu_2^2(\nu_1+\nu_2-2)}{\nu_1(\nu_2-2)^2(\nu_2-4)}$ for $\nu_2>4$ (does not exist for $\nu_2\le 4$). kth moment: $$E[X^k]=\frac{\Gamma(\frac{1}{2}\nu_1+k)\Gamma(\frac{1}{2}\nu_2-k)}{\Gamma(\frac{1}{2}\nu_1)\Gamma(\frac{1}{2}\nu_2)}\left(\frac{\nu_2}{\nu_1}\right)^k,\quad 2k<\nu_2$$	*F-distribution.* B is the beta function defined in (9.61). ν_1, ν_2 are the degrees of freedom for the numerator and denominator, respectively.
34.8	$$f(x)=\begin{cases}\dfrac{\lambda^n x^{n-1}e^{-\lambda x}}{\Gamma(n)}, & x>0\\ 0, & x\le 0\end{cases};\quad n,\ \lambda>0.$$ Mean: $E[X]=n/\lambda$. Variance: $\operatorname{var}[X]=n/\lambda^2$. Moment generating function: $[\lambda/(\lambda-t)]^n,\ t<\lambda$. Characteristic function: $[\lambda/(\lambda-it)]^n$.	*Gamma distribution.* Γ is the gamma function defined in (9.53). For $n=1$ this is the exponential distribution.