Chapter 4

Partial derivatives

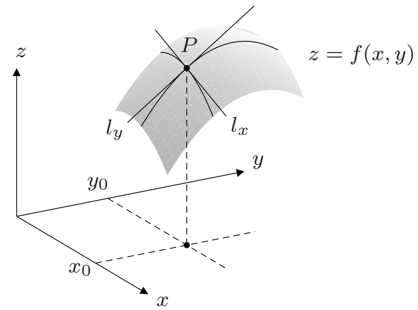
4.1 If $z = f(x_1, \ldots, x_n) = f(\mathbf{x})$, then

$$\frac{\partial z}{\partial x_i} = \frac{\partial f}{\partial x_i} = f_i'(\mathbf{x}) = D_{x_i} f = D_i f$$

all denote the derivative of $f(x_1, \ldots, x_n)$ with respect to x_i when all the other variables are held constant.

Definition of the *partial derivative*. (Other notations are also used.)

4.2



Geometric interpretation of the partial derivatives of a function of two variables, $z = f(x, y)$: $f_1'(x_0, y_0)$ is the slope of the tangent line l_x and $f_2'(x_0, y_0)$ is the slope of the tangent line l_y.

4.3 $$\frac{\partial^2 z}{\partial x_j \partial x_i} = f_{ij}''(x_1, \ldots, x_n) = \frac{\partial}{\partial x_j} f_i'(x_1, \ldots, x_n)$$

Second-order partial derivatives of $z = f(x_1, \ldots, x_n)$.

4.4 $$\frac{\partial^2 f}{\partial x_j \partial x_i} = \frac{\partial^2 f}{\partial x_i \partial x_j}, \quad i, j = 1, 2, \ldots, n$$

Young's theorem, valid if the two partials are continuous.

4.5 $f(x_1, \ldots, x_n)$ is said to be *of class* C^k, or simply C^k, in the set $S \subset \mathbb{R}^n$ if all partial derivatives of f of order $\leq k$ are continuous in S.

Definition of a C^k function. (For the definition of continuity, see (12.14).)

4.6 $z = F(x, y),\ x = f(t),\ y = g(t) \ \Rightarrow$

$$\frac{dz}{dt} = F_1'(x, y)\frac{dx}{dt} + F_2'(x, y)\frac{dy}{dt}$$

A *chain rule*.

4.7 If $z = F(x_1, \ldots, x_n)$ and $x_i = f_i(t_1, \ldots, t_m)$, $i = 1, \ldots, n$, then for all $j = 1, \ldots, m$

$$\frac{\partial z}{\partial t_j} = \sum_{i=1}^{n} \frac{\partial F(x_1, \ldots, x_n)}{\partial x_i} \frac{\partial x_i}{\partial t_j}$$

The chain rule. (General case.)

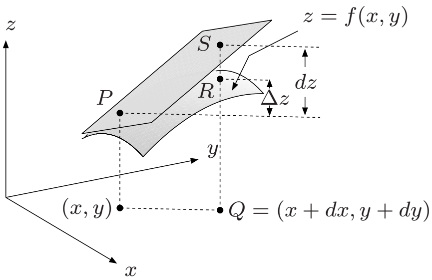
4.8 If $z = f(x_1, \ldots, x_n)$ and $dx_1, \ldots, dx_n$ are arbitrary numbers,

$$dz = \sum_{i=1}^{n} f_i'(x_1, \ldots, x_n)\, dx_i$$

is the *differential* of z.

Definition of the differential.

4.9



Geometric illustration of the definition of the differential for functions of two variables. It also illustrates the approximation $\Delta z \approx dz$ in (4.10).

4.10 $\Delta z \approx dz$ when $|dx_1|, \ldots, |dx_n|$ are all small, where

$$\Delta z = f(x_1 + dx_1, \ldots, x_n + dx_n) - f(x_1, \ldots, x_n)$$

A useful approximation, made more precise for differentiable functions in (4.11).

4.11 f is *differentiable* at $\mathbf{x}$ if $f_i'(\mathbf{x})$ all exist and there exist functions $\varepsilon_i = \varepsilon_i(dx_1, \ldots, dx_n)$, $i = 1, \ldots, n$, that all approach zero as dx_i all approach zero, and such that

$$\Delta z - dz = \varepsilon_1\, dx_1 + \cdots + \varepsilon_n\, dx_n$$

Definition of differentiability.

4.12 If f is a C^1 function, i.e. it has continuous first order partials, then f is differentiable.

An important fact.

4.13

$$d(af + bg) = a\, df + b\, dg \quad (a \text{ and } b \text{ constants})$$
$$d(fg) = g\, df + f\, dg$$
$$d(f/g) = (g\, df - f\, dg)/g^2$$
$$dF(u) = F'(u)\, du$$

Rules for differentials. f and g are differentiable functions of $x_1, \ldots, x_n$, F is a differentiable function of one variable, and u is any differentiable function of $x_1, \ldots, x_n$.