Chapter 6

Systems of equations

6.1
$$\begin{aligned} f_1(x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_m) &= 0 \\ f_2(x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_m) &= 0 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots & \\ f_m(x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_m) &= 0 \end{aligned}$$

A general system of equations with n *exogenous variables*, $x_1, \ldots, x_n$, and m *endogenous variables*, $y_1, \ldots, y_m$.

6.2
$$\frac{\partial \mathbf{f}(\mathbf{x}, \mathbf{y})}{\partial \mathbf{y}} = \begin{pmatrix} \dfrac{\partial f_1}{\partial y_1} & \cdots & \dfrac{\partial f_1}{\partial y_m} \\ \vdots & \ddots & \vdots \\ \dfrac{\partial f_m}{\partial y_1} & \cdots & \dfrac{\partial f_m}{\partial y_m} \end{pmatrix}$$

The *Jacobian matrix* of $f_1, \ldots, f_m$ with respect to $y_1, \ldots, y_m$.

6.3 Suppose $f_1, \ldots, f_m$ are C^k functions in a set A in $\mathbb{R}^{n+m}$, let $(\mathbf{x}^0, \mathbf{y}^0) = (x_1^0, \ldots, x_n^0, y_1^0, \ldots, y_m^0)$ be a solution to (6.1) in the interior of A. Suppose also that the determinant of the Jacobian matrix $\partial \mathbf{f}(\mathbf{x}, \mathbf{y})/\partial \mathbf{y}$ in (6.2) is different from 0 at $(\mathbf{x}^0, \mathbf{y}^0)$. Then (6.1) defines $y_1, \ldots, y_m$ as C^k functions of $x_1, \ldots, x_n$ in some neighborhood of $(\mathbf{x}^0, \mathbf{y}^0)$, and the Jacobian matrix of these functions with respect to $\mathbf{x}$ is

$$\frac{\partial \mathbf{y}}{\partial \mathbf{x}} = \left(\frac{\partial \mathbf{f}(\mathbf{x}, \mathbf{y})}{\partial \mathbf{y}}\right)^{-1} \frac{\partial \mathbf{f}(\mathbf{x}, \mathbf{y})}{\partial \mathbf{x}}$$

The *general implicit function theorem.* (It gives sufficient conditions for system (6.1) to define the endogenous variables $y_1, \ldots, y_m$ as differentiable functions of the exogenous variables $x_1, \ldots, x_n$. (For the case $n = m = 1$, see (4.17).)

6.4
$$\begin{aligned} f_1(x_1, x_2, \ldots, x_n) &= 0 \\ f_2(x_1, x_2, \ldots, x_n) &= 0 \\ \cdots\cdots\cdots\cdots\cdots & \\ f_m(x_1, x_2, \ldots, x_n) &= 0 \end{aligned}$$

A general system of m equations and n variables.

6.5 System (6.4) has k *degrees of freedom* if there is a set of k of the variables that can be freely chosen such that the remaining $n - k$ variables are uniquely determined when the k variables have been assigned specific values. If the variables are restricted to vary in a set S in $\mathbb{R}^n$, the system has k *degrees of freedom in* S.

Definition of degrees of freedom for a system of equations.

6.6 To find the number of degrees of freedom for a system of equations, count the number, n, of variables and the number, m, of equations. If $n > m$, there are $n - m$ degrees of freedom in the system. If $n < m$, there is, in general, no solution of the system.

The "counting rule". This is a rough rule which is *not* valid in general.

6.7 If the conditions in (6.3) are satisfied, then system (6.1) has n degrees of freedom.

A precise (local) counting rule.

6.8
$$\mathbf{f}'(\mathbf{x}) = \begin{pmatrix} \dfrac{\partial f_1(\mathbf{x})}{\partial x_1} & \dots & \dfrac{\partial f_1(\mathbf{x})}{\partial x_n} \\ \vdots & & \vdots \\ \dfrac{\partial f_m(\mathbf{x})}{\partial x_1} & \dots & \dfrac{\partial f_m(\mathbf{x})}{\partial x_n} \end{pmatrix}$$

The *Jacobian matrix* of $f_1, \dots, f_m$ with respect to $x_1, \dots, x_n$, also denoted by $\partial \mathbf{f}(\mathbf{x})/\partial \mathbf{x}$.

6.9 If $\mathbf{x}^0 = (x_1^0, \dots, x_n^0)$ is a solution of (6.4), $m \leq n$, and the rank of the Jacobian matrix $\mathbf{f}'(\mathbf{x})$ is equal to m, then system (6.4) has $n - m$ degrees of freedom in some neighborhood of $\mathbf{x}^0$.

A precise (local) counting rule. (Valid if the functions $f_1, \dots, f_m$ are C^1.)

6.10 The functions $f_1(\mathbf{x}), \dots, f_m(\mathbf{x})$ are *functionally dependent* in an open set A in $\mathbb{R}^n$ if there exists a real-valued C^1 function F defined on an open set containing

$$S = \{(f_1(\mathbf{x}), \dots, f_m(\mathbf{x})) : \mathbf{x} \in A\}$$

such that

$$F(f_1(\mathbf{x}), \dots, f_m(\mathbf{x})) = 0 \quad \text{for all } \mathbf{x} \text{ in } A$$

and $\nabla F \neq \mathbf{0}$ in S.

Definition of functional dependence.

6.11 If $f_1(\mathbf{x}), \dots, f_m(\mathbf{x})$ are functionally dependent in an open set $A \subset \mathbb{R}^n$, then the rank of the Jacobian matrix $\mathbf{f}'(\mathbf{x})$ is less than m for all $\mathbf{x}$ in A.

A necessary condition for functional dependence.

6.12 If the equation system (6.4) has solutions, and if $f_1(\mathbf{x}), \dots, f_m(\mathbf{x})$ are functionally dependent, then (6.4) has at least one redundant equation.

A sufficient condition for the counting rule to fail.