Chapter 8

Series. Taylor's formula

8.1	$\sum_{i=0}^{n-1}(a+id) = na + \frac{n(n-1)d}{2}$	Sum of the first n terms of an *arithmetic series.*
8.2	$a + ak + ak^2 + \cdots + ak^{n-1} = a\,\frac{1-k^n}{1-k},\ k \neq 1$	Sum of the first n terms of a *geometric series.*
8.3	$a + ak + \cdots + ak^{n-1} + \cdots = \frac{a}{1-k}$ if $\lvert k\rvert < 1$	Sum of an infinite geometric series.
8.4	$\sum_{n=1}^{\infty} a_n = s$ means that $\lim_{n\to\infty}\sum_{k=1}^{n} a_k = s$	Definition of the *convergence* of an infinite series. If the series does not converge, it *diverges*.
8.5	$\sum_{n=1}^{\infty} a_n$ converges $\Rightarrow$ $\lim_{n\to\infty} a_n = 0$	A necessary (but NOT sufficient) condition for the convergence of an infinite series.
8.6	$\lim_{n\to\infty}\left\lvert\frac{a_{n+1}}{a_n}\right\rvert < 1 \Rightarrow \sum_{n=1}^{\infty} a_n$ converges	The *ratio test*.
8.7	$\lim_{n\to\infty}\left\lvert\frac{a_{n+1}}{a_n}\right\rvert > 1 \Rightarrow \sum_{n=1}^{\infty} a_n$ diverges	The *ratio test*.
8.8	If $f(x)$ is a positive-valued, decreasing, and continuous function for $x \geq 1$, and if $a_n = f(n)$ for all integers $n \geq 1$, then the infinite series and the improper integral $\sum_{n=1}^{\infty} a_n$ and $\int_1^{\infty} f(x)\,dx$ either both converge or both diverge.	The *integral test*.

8.9	If $0 \le a_n \le b_n$ for all n, then • $\sum a_n$ converges if $\sum b_n$ converges. • $\sum b_n$ diverges if $\sum a_n$ diverges.	The *comparison test.*
8.10	$\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^p}$ is convergent $\iff p > 1$	An important result.
8.11	A series $\sum_{n=1}^{\infty} a_n$ is said to *converge absolutely* if the series $\sum_{n=1}^{\infty} \lvert a_n \rvert$ converges.	Definition of absolute convergence. $\lvert a_n \rvert$ denotes the absolute value of a_n.
8.12	Every absolutely convergent series is convergent, but not all convergent series are absolutely convergent.	A convergent series that is not absolutely convergent, is called *conditionally convergent.*
8.13	If a series is absolutely convergent, then the sum is independent of the order in which terms are summed. A conditionally convergent series can be made to converge to any number (or even diverge) by suitable rearranging the order of the terms.	Important results on the convergence of series.
8.14	$f(x) \approx f(a) + f'(a)(x-a) \qquad (x \text{ close to } a)$	*First-order (linear)* approximation about $x = a$.
8.15	$f(x) \approx f(a) + f'(a)(x-a) + \frac{1}{2}f''(a)(x-a)^2 \qquad (x \text{ close to } a)$	*Second-order (quadratic)* approximation about $x = a$.
8.16	$\displaystyle f(x) = f(0) + \frac{f'(0)}{1!}x + \cdots + \frac{f^{(n)}(0)}{n!}x^n + \frac{f^{(n+1)}(\theta x)}{(n+1)!}x^{n+1}, \quad 0 < \theta < 1$	*Maclaurin's formula.* The last term is Lagrange's error term.
8.17	$\displaystyle f(x) = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \cdots$	The *Maclaurin series* for $f(x)$, valid for those x for which the error term in (8.16) tends to 0 as n tends to ∞.
8.18	$\displaystyle f(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \frac{f^{(n+1)}(a+\theta(x-a))}{(n+1)!}(x-a)^{n+1}, \quad 0 < \theta < 1$	*Taylor's formula.* The last term is Lagrange's error term.