Replicable Functions: An Introduction

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Summary. We survey the theory of replicable functions and its ramifications from number theory to physics.

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1 Introduction

We survey the theory of replicable functions and matters of related interest. These functions were introduced in monstrous moonshine [4], characterizing the principal moduli attached to the conjugacy classes of the monster simple group, $\mathbb{M}$. It is not surprising that replicable functions, being related to moonshine and the monster, have a wide range of connections to other fields of mathematics and physics which remain to be fathomed. Indeed, moonshine has been described as 21st century mathematics in the 20th century. Having arrived, we can survey the past 25 years with some satisfaction but there is much remaining to be clarified and put into an appropriate context. The field is amazingly fertile: there are connections with several aspects of mathematical physics and number theory, and one finds classical and modern themes continually coming into play. We explain a few of these connections, some of which are presented here for the first time.

It is simplest to define replicable functions through the Faber polynomials to which the next section is devoted. We then provide examples related to classical themes such as Chebyshev polynomials and Hecke operators. Later sections will deal with the automorphic aspect of the replicable functions, links with the Schwarz derivative, the characterization of the monstrous moonshine functions, the exceptional affine correspondences, class numbers, and the soliton equations and their $\tau-$function from the 2D Toda hierarchy.

2 Faber polynomials

The Faber polynomials [9] originated in approximation theory in 1903 and are central to the theory of replicable functions. We define them in a formal way, leaving complex analysis and Riemann mappings for later in section 3 and section 12.

Let $f$ be a function given by the expansion

$$f(q) = \frac{1}{q} + \sum_{n=1}^{\infty} a_n q^n,$$  \hspace{1cm} (2.1)

where we take $q = \exp(2\pi i z), z \in \mathcal{H}$, the upper half–plane. Throughout, we interpret derivatives of $f$ with respect to its argument. We initially assume that the coefficients $a_n \in \mathbb{C}$, and we choose the constant term to be zero. For each $n \in \mathbb{N}$, there exists a unique monic polynomial $F_n$ such that

$$F_n(f(q)) = \frac{1}{q^n} + O(q) \quad \text{as} \quad q \to 0.$$

In fact $F_n = F_{n,f}$ depends on the coefficients of $f$, but we denote it simply by $F_n$ when there is no confusion. It can be shown that the Faber polynomials are given by the generating series