The No-Arbitrage Property under a Change of Numéraire (1995)

Abstract. For a price process that has an equivalent risk neutral measure, we investigate if the same property holds when the numéraire is changed. We give necessary and sufficient conditions under which the price process of a particular asset — which should be thought of as a different currency — can be chosen as new numéraire. The result is related to the characterisation of attainable claims that can be hedged. Roughly speaking: the asset representing the new currency is a reasonable investment (in terms of the old currency) if and only if the market does not permit arbitrage opportunities in terms of the new currency as numéraire. This rough but economically meaningful idea is given a precise content in this paper. The main ingredients are a duality relation as well as a result on maximal elements. The paper also generalises results previously obtained by Jacka, Ansel-Stricker and the authors.

11.1 Introduction

In this paper we deal with the change of numéraire problem. Let us assume that a $d$-dimensional process $S$ describes the price of $d$ assets in a fixed chosen currency unit. If e.g. the currency unit is changed, the price process $S$ will be multiplied by the exchange ratio describing the old currency in function of the new one. We shall give examples showing that the no-arbitrage property of the process $S$ may depend on the choice of numéraire. Such an example was already given in [DS94a]. The question now arises when the value of an asset or more generally of a portfolio, can be used as a new numéraire without destroying the no-arbitrage property. Of course this will depend on the kind of no-arbitrage we use. We will give precise definitions further in the paper but for the moment let us assume (oversimplifying things) that no-


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It turns out that the problem is related to the characterisation of those contingent claims that can be hedged. This topic was studied by Jacka [J92] and Ansel-Stricker [AS 94]. These authors use the $H^1$-BMO duality. We will give a measure independent characterisation in terms of maximal elements of attainable claims. These elements were already used, as a technical device, in Chap. 9. The proofs of the theorems below use these results as well as an extension of a duality relation from Delbaen [D 92].

The technique of a change of numéraire together with the change of the risk neutral measure was used by El Karoui, Geman and Rochet [EGR 95] and Jamshidian [J 87] to facilitate calculations of prices of contingent claims. The results of this paper can also be used to build consistent models of exchange rates of currencies. In this case the discounting procedure depends on the currency since the interest rate in different currencies will be different. We refer to Delbaen-Shirakawa [DSh 96] for details.

The rest of this section is devoted to the introduction of the basic notation. Sect. 11.2 recalls known facts from arbitrage theory. In Sect. 11.3 we extend the duality equality and relate it to properties of maximal elements. Sect. 11.4 finally contains the main theorem on the change of numéraire and the application to the theory of hedgeable elements.

The setup in this paper is the usual setup in mathematical finance. A probability space $(\Omega, \mathcal{F}, P)$ with a filtration $(\mathcal{F}_t)_{0 \leq t}$ is given. In order to cover the most general case, the time set is supposed to be $\mathbb{R}_+$. The filtration is assumed to satisfy the “usual conditions”, i.e. it is right continuous and $\mathcal{F}_0$ contains all null sets of $\mathcal{F}$. A price process $S$, describing the evolution of the discounted price of $d$ assets, is defined on $\mathbb{R}_+ \times \Omega$ and takes values in $\mathbb{R}^d$. In order to use the results of Chap. 9, we suppose that the process $S$ is locally bounded. This assumption is fairly general, in particular it covers the case of continuous price processes. As shown under a wide range of hypotheses, the assumption that $S$ is a semi-martingale follows from arbitrage considerations. We can therefore assume that the process $S$ is a semi-martingale. Since it is also locally bounded it is a special semi-martingale. Stochastic integration is used to describe outcomes of investment strategies. When dealing with processes in dimension higher than 1 it is understood that vector stochastic integration is used. We refer to Protter [P 90] and Jacod [J 79] for details on these matters.

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Note added in this reprint: The idea of changing the numéraire can be traced back to the work of Margrabe [M 78a, M 78b]