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Mindlin's Electro-Elastic Coupling
due to Polarization Gradient

2.1 Toupin's alternative formulation of the theory of piezoelectricity

(a) On the physical meaning of the polarization vector

As it turned out, the electrostriction is not at all the only alternative to the piezoelectric coupling. It was already mentioned in the Introduction about the much less traditional and really fruitful idea of the polarization gradient [103 - 104], which provides a new linear mechanism of electro-elastic coupling. However, in order to start with this new ideology we have to revise the above classical theory of linear piezoelectricity and to introduce some formalism, which allows explicit involving of the vector of polarization \( \mathbf{P} \), instead of the electric displacement \( \mathbf{D} \), in constitutive relations and equations of motion.

A dielectric body placed into an external electric field \( \mathbf{E}_0 \) reacts on the latter by a polarization of the molecules, which is macroscopically characterized by the distributed dipole moment \( \mathbf{P} \) per unit volume. The vector \( \mathbf{P} \) is customary named the polarization vector. This polarization of the medium creates the so-called depolarization field \( \mathbf{E}^d \), which weakens the initial electric field \( \mathbf{E}^0 \), so that the resulting electric field

\[
\mathbf{E} = \mathbf{E}^0 + \mathbf{E}^d
\]

(2.1)

is always less than \( \mathbf{E}^0 \). Of course, if the dielectric body came to its place together with its own sources (e.g., charges in the bulk or at the surface) creating the additional field \( \mathbf{E}^s \), the resulting electric field will be

\[
\mathbf{E} = \mathbf{E}^0 + \mathbf{E}',
\]

(2.2)

where \( \mathbf{E}' = \mathbf{E}^d + \mathbf{E}^s \) (compare with (1.51)). It is essential that the insertion of the dielectric body into the field \( \mathbf{E}^0 \) of external sources perturbs this field not only in the bulk of the body but, as a rule, also outside of it. That is why the electrodynamics of condensed media ordinarily deals with a resulting field \( \mathbf{E} \),
rather than with its constitutive components (2.2). Instead of decomposing
electric field (2.2) it has turned out to be much more convenient to combine
two physically meaningful quantities, the electric field $E$ and the polarization
vector $P$, composing a formal but very convenient vector-function, the so-called
electric displacement
\[ D = \varepsilon_0 E + P, \] (2.3)
where $\varepsilon_0$ is the electric permittivity of the vacuum. As is well known, in terms
of the pair, $E$ and $D$, together with analogous magnetic pair, $B$ and $H$ (in this
case $B$ is a physical quantity and $H$ is a formal function), Maxwell’s equations
and boundary conditions look very compact and convenient (see, e.g., (1.1),
(1.41), (1.58), (1.60)).

(b) Toupin’s potential $U^L$ of polarization energy

Paradoxically following to the opposite tendency of dividing of the electrical
characteristics into vacuum and polarization contributions, Toupin [105] sug-
gested counting off the energy $U$ of the dielectric body from the energy of the
resulting electric field in the vacuum $U_0 = \frac{1}{2} \varepsilon_0 E^2$, so that
\[ U = \frac{1}{2} \varepsilon_0 E^2 + U^L(\beta, P). \] (2.4)
For simplicity we omit here a thermal variable. In accordance with eq. (1.12)
under the adiabatic conditions and in the absence of the magnetic field the
rate of change of the energy density $U(\beta, D)$ is determined by
\[ \dot{U} = \sigma_{ij} \dot{\beta}_{ij} + E_i \dot{D}_i. \] (2.5)
This gives
\[ \sigma_{ij} = \left( \frac{\partial U}{\partial \beta_{ij}} \right)_D, \quad E_i = \left( \frac{\partial U}{\partial D_i} \right)_\beta. \] (2.6)
Now let us express the potential $U^L(\beta, P)$ using eqs. (2.4) and (2.3):
\[ U^L(\beta, P) = U[\beta, \varepsilon_0 E(\beta, P) + P] - \frac{1}{2} \varepsilon_0 [E(\beta, P)]^2, \] (2.7)
and calculate the following derivatives
\[ \left( \frac{\partial U^L}{\partial \beta_{ij}} \right)_P = \left( \frac{\partial U}{\partial \beta_{ij}} \right)_D + \varepsilon_0 \left( \frac{\partial U}{\partial D_i} \right)_\beta \left( \frac{\partial E_i}{\partial \beta_{ij}} \right)_P - \varepsilon_0 E_i \left( \frac{\partial E_i}{\partial \beta_{ij}} \right)_P, \] (2.8)
\[ \left( \frac{\partial U^L}{\partial P_i} \right)_\beta = \left( \frac{\partial U}{\partial D_i} \right)_\beta \left[ \varepsilon_0 \left( \frac{\partial E_i}{\partial P_i} \right)_\beta + \delta_{ii} \right] - \varepsilon_0 E_i \left( \frac{\partial E_i}{\partial P_i} \right)_\beta. \] (2.9)
In (2.9) $\delta_{ii}$ is the Kronecker delta. In view of eqs. (2.6), the latter derivatives
are reduced to the constitutive relations