Cofree Coalgebras for Signature Morphisms*

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Abstract. The paper investigates the construction of cofree coalgebras for 'unsorted signature morphisms'. Thanks to the perfect categorical duality between the traditional concept of equations and the concept of coequations developed in [14] we can fully take profit of the methodological power of Category Theory [2] and follow a clean three step strategy: Firstly, we analyse the traditional BIRKHOFF construction of free algebras and reformulate it in a systematic categorical way. Then, by dualizing the BIRKHOFF construction, we obtain, in a second step, corresponding results for cofree coalgebras. And, thirdly, we will interpret the new "abstract" categorical results in terms of more familiar concept. The analysis of a sample cofree construction will provide, finally, some suggestions concerning the potential rôle of cofree coalgebras in System Specifications.

1 Introduction

It is an old observation that unsorted signatures used in Universal Algebra and Algebraic Specifications can be coded by functors $\mathcal{F} : \mathbf{Set} \to \mathbf{Set}$. $\mathcal{F}$-algebras are given in this setting by a carrier A and a map $\alpha : \mathcal{F}(A) \to A$. A corresponding unsorted signature morphism is modeled by a natural transformation $\tau : \mathcal{F} \Rightarrow \mathcal{G}$ and gives rise to a forgetful functor $U_\tau : \mathbf{Alg}(\mathcal{G}) \to \mathbf{Alg}(\mathcal{F})$, where $\mathbf{Alg}(\mathcal{F})$ denotes the category of all $\mathcal{F}$-algebras and all homomorphisms between them.

Free algebras w.r.t. U_τ and the corresponding free functor $T_\tau : \mathbf{Alg}(\mathcal{F}) \to \mathbf{Alg}(\mathcal{G})$, i.e., the functor left-adjoint to U_τ, play an important rôle in Algebraic Specifications, especially for parametrization, structuring, and modularization [2–4, 8, 9].

On the other hand, it has become evident, in the last few years, that coalgebras, the categorical dual of algebras, provide a unifying framework for formal specification of dynamical and behavioural aspects of systems [6, 7, 10, 15]. $\mathcal{F}$-coalgebras are given by a carrier A and a (reversed) map $\alpha : A \to \mathcal{F}(A)$. An unsorted signature morphism $\tau : \mathcal{F} \Rightarrow \mathcal{G}$ gives rise to a functor $U^c_\tau : \mathbf{Alg}^c(\mathcal{F}) \to \mathbf{Alg}^c(\mathcal{G})$ between the corresponding categories of coalgebras. Those "co-forgetful" functors have been, e.g., successfully used in establishing a hierarchy of probabilistic system types [1]. In contrast the corresponding cofree functors have not been investigated and applied up to now, as far as we know.

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But, taking into account the importance of free algebras in Algebraic Specifications, we should be curious about the rôle of cofree coalgebras in structuring and modularizing (dynamic) System Specifications.

In the paper we take a first step and investigate the construction of cofree coalgebras for unsorted signature morphisms. Due to the perfect categorical duality between the traditional concept of equations and the concept of coequations developed in [14] we can fully take profit of the methodological power of Category Theory [2] and follow a clean three step strategy: The first, most demanding, step will be to analyse the traditional BIRKHOFF construction of free algebras and to develop a systematic categorical description of this construction. In a second, easy step we will dualize the categorical construction in a quite formal way. And, in a third step, we will try to interpret the dual construtions and results in terms of known concepts, i.e., in terms of concepts we are familiar with because of our today education.

The analysis of a sample (co)free construction will, hopefully, give some hints for a future, more comprehensive investigation of the potential rôle of cofree coalgebras in System Specification. We will close the paper with a short discussion of possible generalizations and extensions of the results presented here.

2 Sets, Algebras, Termalgebras, and Equations

We summarize the necessary concepts, constructions, and results from [14].

A *(generalized) subset* (S, i)[1] of a set A is a set S together with a mono (injective map) $i : S \to A$. We write $(S_1, i_1) \subseteq_A (S_2, i_2)$ (or simply $(S_1, i_1) \subseteq (S_2, i_2)$), if there is a map $m : S_1 \to S_2$ such that $i_1 = i_2 \circ m$, and we write $(S_1, i_1) \cong_A (S_2, i_2)$ in case $(S_1, i_1) \subseteq_A (S_2, i_2)$ and $(S_2, i_2) \subseteq_A (S_1, i_1)$.Note, that S_1 and S_2 are isomorphic if $(S_1, i_1) \cong_A (S_2, i_2)$.

The category **Set** has all limits thus we can construct for any family $\mathcal{S} = ((S_j, i_j) \mid j \in J)$ of subsets of A the multiple pullback $(\bigcap \mathcal{S} \xrightarrow{m_j} S_j \mid j \in J)$. The m_j are mono since multiple pullbacks preserve mono's in any category. In such a way, we obtain a new subset $(\bigcap \mathcal{S}, i_\cap)$ of A with $i_\cap = i_j \circ m_j$, i.e., $(\bigcap \mathcal{S}, i_\cap) \subseteq_A (S_j, i_j)$, for all $j \in J$, called the *intersection* of $\mathcal{S}$.

Using intersection we can define for any map $f : A \to B$ the *image of A under f* : We build the intersection of all subsets (S, i) of B with $f = i \circ l$ for some map $l : A \to S$ and obtain a subset $(f(A), m_f)$ of B and a map $e_f : A \to f(A)$ with $f = m_f \circ e_f$. In **Set** holds the *axiom of choice*: Every epi $e : A \to B$ in **Set** is a *split epi*, i.e., there exists at least one $r : B \to A$ such that $e \circ r = id_B$. This axiom ensures that the image construction provides an epi-mono factorization:

Proposition 1. *Any map $f : A \to B$ in* **Set** *can be factorized as $f = m_f \circ e_f$ with $m_f : f(A) \to B$ a mono and $e_f : A \to f(A)$ a (split) epi.*

A *relation* between two sets A and B is a subset (R, i_R) of the product $A \times B$, where we take here for $A \times B$ the cartesian product $\{(a, b) \mid a \in A, b \in B\}$.

[1] Note, that the position of S indicates that S is the domain of i.