Uncovering and Reducing Hidden Combinatorics in Guigues-Duquenne Bases

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Abstract. Mannila and Räihä [5] have shown that minimum implicational bases can have an exponential number of implications. Aim of our paper is to understand how and why this combinatorial explosion arises and to propose mechanisms which reduce it.

Keywords: Guigues-Duquenne base, closure systems, clone attributes.

1 Introduction

One of the most important open problems in formal concept analysis is the generation of a minimum implicational base from a context. This problem has two major practical difficulties.

First, there is no known polynomial time algorithm which computes such a base. This is still an open problem and many ongoing researches try to classify this problem for particular cases (for instance, finding the keys of a multi-valued context [2]). The other critical problem is the size of the result. A minimum implicational base might have an exponential size (see Mannila and Räihä [5]).

In this paper, we explain why such combinatorial explosion arises and then try to reduce it. For this study, we consider only Guigues-Duquenne bases since they are well defined.

In the example given by Mannila and Räihä, one can notice that some attributes play similar roles in the pseudo-closed sets, i.e. some pseudo-closed sets can be obtained from others by simply exchanging one attribute by another one. We say that a and b are *P-clone* attributes if all pseudo-closed sets containing attribute b can be obtained by exchanging a for attribute b in all pseudo-closed sets containing attribute a, and reciprocally.

We believe that the combinatorial explosion of the Guigues-Duquenne base is due to the presence of P-clone attributes. Aim of our ongoing work is either to prove or invalidate this belief. In this paper we present some results which could help in achieving this aim.

Medina and Nourine [6] introduced the notion of clone attributes, which is a relaxed definition of P-clone attributes. Indeed, clone attributes are attributes

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having a similar role on closed sets[1] rather than on pseudo-closed sets. With this notion, it has been shown that the combinatorial explosion of Mannila and Räihä example is due to clone attributes. Moreover, a clone reduction operator which drastically reduces the size of the Guigues-Duquenne base has been proposed. The Mannila and Räihä example reduces to only one implication on its clone-free reduced context.

However, in spite of this reduction, one can easily find a new example of a base with exponential size. In this example, combinatorial explosion is due to the presence of implications having a single attribute as premise. We thus propose a new context transformation operator, called *atomization*, which computes a new context that preserves pseudo-closed sets having more than one attribute in their premise. We thus obtain a new relaxed definition of P-clone attributes: the *A-clone* attributes which are the clone attributes present in an *atomized* context. We do not know if there exists an example of *A-clone free context* with an exponential minimum base. This is still an open problem.

2 Notations, Definitions and Main Problem Statement

2.1 Definitions

In this paper, minimum implicational bases are noted by Σ and are supposed to be in the Guigues-Duquenne form. We abusively use the notation $\mathcal{F}$ to denote a closure system and its corresponding lattice. The classical closure operator (Galois operator) over a context $R = (J, M, I)$ is noted by $\overline{}$. Thus, the closure of x is noted $\overline{x}$.

We suppose that small letters are used to represent attributes of the context and two attributes are not identical in the context; i.e. for any pair of attributes (a, b), $\overline{a} \neq \overline{b}$. We denote by J the set of attributes and M the objects of the context $R = (J, M, I)$. We consider that objects present in the context are the meet-irreducible closed sets of the closure system associated to R. A closed set is said to be meet-irreducible in a closure system if it has exactly one cover.

We briefly recall here the main property of the Guigues-Duquenne base.

Definition 1. *Quasi-closed and Pseudo-closed set*
Let $\mathcal{F}$ be a closure system and $\overline{}$ the closure operator associated to $\mathcal{F}$. Let $P \in 2^J$ and $P \notin \mathcal{F}$.

- *P is a Quasi-closed set iff for all $Q \subset P$, $\overline{Q} \subset P$ or $\overline{Q} = \overline{P}$.*
- *A quasi-closed set P is a pseudo-closed set if there is no quasi-closed set $Q \subset P$ with $\overline{Q} = \overline{P}$.*

Theorem 1. *Duquenne-Guigues base [1].*
Let $\mathcal{F}$ be a closure system. The set $\Sigma_{\mathcal{F}} = \{P \rightarrow \overline{P} \backslash P \mid P \text{ is pseudo closed}\}$ is an implicational base of $\mathcal{F}$ and has a minimum number of implications.

[1] This idea was used in Ganter [3] to generate closed sets under symmetry, i.e. generate only one closed set in each class of closed sets.