Crisply Generated Fuzzy Concepts

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Abstract. In formal concept analysis of data with fuzzy attributes, both the extent and the intent of a formal (fuzzy) concept may be fuzzy sets. In this paper we focus on so-called crisply generated formal concepts. A concept $\langle A, B\rangle \in \mathcal{B}(X, Y, I)$ is crisply generated if $A = D^{\downarrow}$ (and so $B = D^{\downarrow\uparrow}$) for some crisp (i.e., ordinary) set $D \subseteq Y$ of attributes (generator). Considering only crisply generated concepts has two practical consequences. First, the number of crisply generated formal concepts is considerably less than the number of all formal fuzzy concepts. Second, since crisply generated concepts may be identified with a (ordinary, not fuzzy) set of attributes (the largest generator), they might be considered "the important ones" among all formal fuzzy concepts. We present basic properties of the set of all crisply generated concepts, an algorithm for listing all crisply generated concepts, a version of the main theorem of concept lattices for crisply generated concepts, and show that crisply generated concepts are just the fixed points of pairs of mappings resembling Galois connections. Furthermore, we show connections to other papers on formal concept analysis of data with fuzzy attributes. Also, we present examples demonstrating the reduction of the number of formal concepts and the speed-up of our algorithm (compared to listing of all formal concepts and testing whether a concept is crisply generated).

1 Problem Setting and Preliminaries

1.1 Problem Setting

Formal concept analysis (FCA) [12] deals with object-attribute data tables (objects and attributes corresponding to table rows and columns, respectively). In the basic setting, attributes are assumed to be binary, i.e. table entries are 1 or 0 according to whether an attribute applies to an object or not. If the attributes under consideration are fuzzy (like "cheap", "expensive"), each table entry contains a truth degree to which an attribute applies to an object. The degrees can be taken from some appropriate scale containing 0 (does not apply at all) and 1 (fully applies) as bounds. The most popular choice is some subinterval of $[0, 1]$, but in general, degrees need not be numbers. A data table with truth degrees can be considered a many-valued context and can be transformed to a binary data table via so-called conceptual scaling [12]. Alternatively, the table with truth degrees can be approached using the apparatus of FCA generalized

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to fuzzy setting (generalization of FCA from the point of view of fuzzy logic). A general discussion about the relationship between conceptual scaling in the sense of FCA and membership functions in the sense of fuzzy set can be found in [21].

In the present paper, we are interested in FCA of data with fuzzy attributes (FCAf) in the framework of fuzzy logic and fuzzy set theory. Probably the first paper on this was [11]. Later on, FCAf was developed by Pollandt [18] and, independently, by the first author of this paper, e.g. [1, 2, 3, 7]. An important aspect of FCA in general is the possibly large number of formal concepts extracted from data. In this paper, we propose and study what we call crisply generated formal fuzzy concepts. These are particular formal fuzzy concepts which can be considered "more important" than the others (non-crisply generated). Considering only crisply generated concepts, the main practical effect is the reduction of the number of formal concepts extracted from data. In the rest of this section, we present preliminaries on fuzzy logic and FCAf. In Section 2 we present our approach and theoretical results. Section 3 contains examples and experiments studying mainly the reduction of the number of extracted concepts.

1.2 Preliminaries

Fuzzy Sets and Fuzzy Logic. We assume basic familiarity with fuzzy logic and fuzzy sets [16, 13, 6]. An element may belong to a fuzzy set in an intermediate degree not necessarily being 0 or 1. Formally, a fuzzy set A in a universe X is a mapping assigning to each $x \in X$ a truth degree $A(x) \in L$ where L is some partially ordered set of truth degrees containing at least 0 (full falsity) and 1 (full truth). L needs to be equipped with logical connectives, e.g. $\otimes$ (fuzzy conjunction), $\rightarrow$ (fuzzy implication), etc. L together with logical connectives forms a structure $\mathbf{L}$ of truth degrees. We assume that $\mathbf{L}$ forms a so-called complete residuated lattice. Recall that a complete residuated lattice [6, 13, 14] is a structure $\mathbf{L} = \langle L, \wedge, \vee, \otimes, \rightarrow, 0, 1\rangle$ such that (1) $\langle L, \wedge, \vee, 0, 1\rangle$ is a complete lattice (with the least element 0, greatest element 1), i.e. a partially ordered set in which arbitrary infima ($\bigwedge$) and suprema ($\bigvee$) exist; (2) $\langle L, \otimes, 1\rangle$ is a commutative monoid, i.e. $\otimes$ is a binary operation satisfying $x \otimes (y \otimes z) = (x \otimes y) \otimes z$, $x \otimes y = y \otimes x$, and $x \otimes 1 = x$; (3) $\otimes, \rightarrow$ satisfy $x \otimes y \leq z$ iff $x \leq y \rightarrow z$. In what follows, $\mathbf{L}$ always denotes a fixed complete residuated lattice.

The most applied set of truth degrees is the real interval $[0, 1]$; with $a \wedge b = \min(a, b)$, $a \vee b = \max(a, b)$, and with three important pairs of fuzzy conjunction and fuzzy implication: Łukasiewicz ($a \otimes b = \max(a + b - 1, 0)$, $a \rightarrow b = \min(1 - a + b, 1)$), minimum ($a \otimes b = \min(a, b)$, $a \rightarrow b = 1$ if $a \leq b$ and $= b$ else), and product ($a \otimes b = a \cdot b$, $a \rightarrow b = 1$ if $a \leq b$ and $= b/a$ else). Another possibility is to take a finite chain $\{a_0 = 0, a_1, \dots, a_n = 1\}$ ($a_0 < \cdots < a_n$) equipped with Łukasiewicz structure ($a_k \otimes a_l = a_{\max(k+l-n,0)}$, $a_k \rightarrow a_l = a_{\min(n-k+l,n)}$) or minimum ($a_k \otimes a_l = a_{\min(k,l)}$, $a_k \rightarrow a_l = a_n$ for $a_k \leq a_l$ and $a_k \rightarrow a_l = a_l$ otherwise).

The set of all fuzzy sets (or $\mathbf{L}$-sets) in X is denoted L^X. For a fuzzy set $A \in L^X$, the 1-cut ${}^1A$ of A is an ordinary set ${}^1A = \{x \in X \mid A(x) = 1\}$. A is called