Complete Subalgebras of Semiconcept Algebras and Protoconcept Algebras

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Abstract. In order to define a negation on formal concepts in Formal Concept Analysis, the more general notions of semiconcepts and protoconcepts were introduced. The theory of the resulting protoconcept and semiconcept algebras is developed in Boolean Concept Logic as a part of Contextual Logic. In this paper it is shown that each complete subalgebra of a semiconcept algebra is itself the semiconcept algebra of an appropriate context. An analogous result holds for the complete subalgebras of protoconcept algebras. These contexts can be obtained from the original context through partitions of the object and the attribute set satisfying certain conditions. Characterizations of the complete subalgebras of semiconcept and protoconcept algebras in terms of contexts, in terms of subsets, and through closed subrelations are given.

1 Introduction

Formal Concept Analysis developed as a mathematical theory of concepts is used succesfully in the area of knowledge representation and knowledge processing. The advantage of this approach lies in the close relation to conceptual human thinking. Yet, while negations of concepts are common in human language and human thinking (e.g. non-smoker, non-profit, non-fiction, NGO), negated concepts generally cannot be represented in concept lattices. Boolean Concept Logic is a theory that extends Formal Concept Analysis by introducing negations of concepts. It is therefore a part of Contextual Logic whose aim is to mathematize the philosophical logic with its doctrines of concepts, judgments, and conclusions (for a brief introduction to Contextual Logic we refer to [Wi00b] or [DK03], an overview over existing theories is given in [KV03]. For more detailed work on Concept Logic see [VW03], [Vo03], [Vo04], recent work on Contextual Judgment Logic can be found in [Da03], [Wi01]).

In Boolean Concept Logic, the negation of a concept is modeled by taking set complements. Since formal concepts consist of two sets, the extent and the intent, we distinguish two kinds of negation: Firstly, the operation $\neg(A,B) := (G\setminus A, (G\setminus A)')$ on the extent side, which will also be called "negation". Secondly, we define an operation $\lrcorner(A,B) := ((M\setminus B)', M\setminus B)$ on the intent side, which will be called "opposition" of a formal concept (A,B) in a context $\mathbb{K}$. This distinction

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	cold	moist	dry	warm
water	×	×		
earth	×		×	
air		×		×
fire			×	×

Fig. 1. The four elements and their attributes in greek philosophy

can already be found in ancient philosophical logic (c.f. [Wi00a]). The negation $\neg$ corresponds to those negations we think of when using words like "non-smoker": In an appropiate context (e.g. with all humans as objects and with at least the attribute "smokes") the word "non-smoker" refers to all objects that are not in the extent of the concept generated by "smokes". The opposition $\lrcorner$ is best illustrated in a context where the attribute set consists of pairs of attributes that mutually exclude each other (i.e. for every such pair $\{m_1, m_2\}$ we have $(g, m_1) \in I \Leftrightarrow (g, m_2) \notin I$ for every object g). The context given in Figure 1, for example, has as attributes the dychotomic pairs "dry" $\leftrightarrow$ "moist" and "cold" $\leftrightarrow$ "warm". The opposition $\lrcorner(\{water\}, \{moist, cold\})$ of the concept generated by water yields the concept $(\{fire\}, \{dry, warm\})$, which is the opposite of this element in ancient greek philosophy (cp. [Wi00a]).

As in general the set complement of an extent (intent) is not an extent (intent) itself, the notion of formal concept is generalized to semiconcepts:

Definition 1. *A* semiconcept *of a formal context* $\mathbb{K} := (G, M, I)$ *is a pair* (A, B) *with* $A \subseteq G$ *and* $B \subseteq M$ *such that* $A' = B$ *or* $B' = A$*. We denote the set of all semiconcepts of a context* $\mathbb{K}$ *by* $\mathfrak{H}(\mathbb{K})$ *and define on* $\mathfrak{H}(\mathbb{K})$ *operations* $\sqcap$, $\sqcup$, $\lrcorner$, $\neg$, $\top$ *and* $\bot$ *by:*

$$
\begin{aligned}
(A_1, B_1) \sqcap (A_2, B_2) &:= (A_1 \cap A_2, (A_1 \cap A_2)') \\
(A_1, B_1) \sqcup (A_2, B_2) &:= ((B_1 \cap B_2)', B_1 \cap B_2) \\
\neg(A, B) &:= (G \setminus A, (G \setminus A)') \\
\lrcorner(A, B) &:= ((M \setminus B)', M \setminus B) \\
\top &:= (G, \emptyset) \\
\bot &:= (\emptyset, M)
\end{aligned}
$$

The set of all semiconcepts of a context $\mathbb{K}$ *together with these operations is called the* semiconcept algebra *of* $\mathbb{K}$ *and denoted by* $\underline{\mathfrak{H}}(\mathbb{K})$.

The operations are called "meet" ($\sqcap$), "join" ($\sqcup$), "negation" ($\neg$), "opposition" ($\lrcorner$), "all" ($\top$) and "nothing" ($\bot$).