Coherence Networks of Concept Lattices: The Basic Theorem

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Abstract. For representing different views and their connections, networks of formal contexts are considered which are coded by so-called *multicontexts*. The coincidences between the network contexts of a multicontext give rise to a *coherence network of concept lattices*. It is the aim of this paper to state and to prove the *Basic Theorem on Coherence Networks of Concept Lattices* as an extension of the Basic Theorem on Concept Lattices.

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1 Introduction

Formal contexts are the basic mathematical structures of *Formal Concept Analysis*. They allow to speak mathematically about *formal objets*, *formal attributes*, and the binary relation which indicates when a formal object *has a* formal attribute. This foundational setting was created for mathematizing concepts and concept hierarchies by so-called *formal concepts* and *concept lattices* of formal contexts (see [Wi82], [GW99]). Since data tables can be mathematized by formal contexts, Formal Concept Analysis has found extensive applications, in particular in data analysis and knowledge processing (cf. [GWW87],[Wi92],[WZ94], [Wi97], [SW00],[Wi00],[Wi02],[Ek04]).

Quite often, the object-attribute-relation is considered under different views so that a representation by a single formal context is not sufficient. This has led to generalizations of the notion of formal context. One approach is based on the mathematization of the ternary relationship that an object has an attribute under a certain condition. This approach, based on *triadic contexts* and derived

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triadic concepts, has been elaborated under the heading *Triadic Concept Analysis* (see [Wi95],[LW95], [Bi98],[WZ00],[DW00]).

The approach we discuss in this paper maintains the vivid dyadic setting, but enables to represent different views and their connections by a network of formal contexts coded in a so-called *multicontext* [Wi96]. In contrast to triadic contexts, multicontexts are composed by formal contexts which may have different object sets and different attribute sets. Those context sets might have some elements in common, where elements could even be objects in one formal context and attributes in another. The coincidences between the context sets of a multicontext give rise to a *coherence network of concept lattices* which are derived from the formal contexts of the multicontext. It is the aim of this paper to state and to prove the so-called *Basic Theorem on Coherence Networks of Concept Lattices* [Dö99] as an extension of the Basic Theorem on Concept Lattices [Wi82] which is stated as follows (cf. [GW99]):

Basic Theorem on Concept Lattices. *Let $\mathbb{K} := (G, M, I)$ be a formal context. Then the set $\underline{\mathfrak{B}}(\mathbb{K})$ of all formal concepts of $\mathbb{K}$ ordered by the subconcept-superconcept-relation is a complete lattice, called the* concept lattice *of (G, M, I), for which infima and suprema can be described in the following way:*

$$\bigwedge_{t\in T}(A_t, B_t) = (\bigcap_{t\in T} A_t, (\bigcup_{t\in T} B_t)^{II}),$$
$$\bigvee_{t\in T}(A_t, B_t) = ((\bigcup_{t\in T} A_t)^{II}, \bigcap_{t\in T} B_t).$$

In general, a complete lattice L is isomorphic to $\underline{\mathfrak{B}}(\mathbb{K})$ if and only if there exist mappings $\tilde{\gamma} : G \longrightarrow L$ and $\tilde{\mu} : M \longrightarrow L$ such that $\tilde{\gamma}G$ is supremum-dense *in L (i.e. $L = \{\bigvee X \mid X \subseteq \tilde{\gamma}G\}$), $\tilde{\mu}M$ is* infimum-dense *in L (i.e. $L = \{\bigwedge X \mid X \subseteq \tilde{\mu}M\}$), and $gIm \iff \tilde{\gamma}g \leq \tilde{\mu}m$ for $g \in G$ and $m \in M$; in particular, $L \cong \underline{\mathfrak{B}}(L, L, \leq)$.*

2 Multicontexts and Coherence Mappings

First we generalize the notion of a formal context to that of a multicontext which can be viewed as a network of formal contexts.

Definition 1. *A* multicontext of signature $\sigma : P \to I^2$*, where I and P are non-empty sets, is defined as a pair (S_I, R_P) consisting of a family $S_I := (S_i)_{i\in I}$ of sets and a family $R_P := (R_p)_{p\in P}$ of binary relations with $R_p \subseteq S_i \times S_j$ if $\sigma p = (i, j)$. A multicontext (S_I, R_P) of signature $\sigma : P \to I^2$ can be understood as a network of formal contexts $\mathbb{K}_p := (S_i, S_j, R_p)$ with $\sigma p = (i, j)$.*

The common elements which emerge in different formal contexts induce a coherence between the concept lattices of those formal contexts. Even with identical sets of objects and sets of attributes, the relation of distinct formal contexts could vary from each other so that one could hardly see any coherence between