Planarity of Lattices

An Approach Based on Attribute Additivity

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Abstract. Popular lattice drawing algorithms do not take planarity into account and find plane diagrams mainly heuristically. We present a characterization of planar lattices based on a theorem of Dushnik and Miller [4] and the “left”-relation introduced by Kelly and Rival [6]. In particular, our work is helpful for drawing plane attribute additive diagrams.

1 Motivation

A lattice is *planar* if it admits a diagram with no edge crossings. There exist algorithms for constructing such plane diagrams (see [3] for an overview), but these do not use the lattice structure and treat the problem as a graph drawing task. Our aim is to automatically construct plane diagrams of planar lattices. Additionally we want to draw them *attribute additively* [7] since this convention provides nice visualizations of lattices.

2 Introduction

Throughout the paper we assume finiteness. For easier notation we use the symbols $\leq$ and $<$ both for lattice order relations and the usual order on $\mathbb{R}$.

2.1 Diagrams of Lattices

A lattice $(\mathfrak{V}, \leq)$ is often represented by a diagram. We draw a small circle for each lattice element and a line for each pair v, w of lattice elements in neighbour relation (i.e. $v < w$ and there is no element z fulfilling $v < z < w$). Lattice diagrams are drawn upward. We define, according to [6]:

Definition 1. *Let* $\underline{\mathfrak{V}} = (\mathfrak{V}, \leq)$ *be a lattice with the neighbour relation* $\prec$*. A* diagram *(or* representation [6]*)* pos$(\underline{\mathfrak{V}})$ *of* $\underline{\mathfrak{V}}$ *is the image of a mapping*

$$\text{pos} : \mathfrak{V} \cup \prec \mapsto \mathbb{R}^2 \cup \mathcal{P}(\mathbb{R}^2)$$

meeting the following conditions for all $v, w, z \in \mathfrak{V}$*.*

B. Ganter and R. Godin (Eds.): ICFCA 2005, LNAI 3403, pp. 391–402, 2005.

1. $\text{pos}\,|_{\mathfrak{V}} : v \mapsto \text{pos}(v) = (x(v), y(v)) \in \mathbb{R}^2$ *is an injection.*
2. *Whenever* $v < w$ *holds then* $y(v) < y(w)$.
3. $\text{pos}\,|_{\prec} : vw \mapsto \{(x_{vw}(y), y) \mid \; y \in [y(v), y(w)]\} \subseteq \mathbb{R}^2$*, where* x_{vw} *is a continuous function with* $x_{vw}(y(v)) = x(v)$ *and* $x_{vw}(y(w)) = x(w)$ *for each pair* $v \prec w$.
4. *If* $\text{pos}(v) \in \text{pos}(wz)$ *holds then* $v = w$ *or* $v = z$.

The elements of $\text{pos}(\mathfrak{V}) := \{\text{pos}(v) \mid \; v \in \mathfrak{V}\}$ *are called* (diagram) points *or* nodes*, the elements of* $\text{pos}(\prec) := \{\text{pos}(vw) \mid \; v, w \in \mathfrak{V}, v \prec w\}$ *are called* diagram edges*.*

Line diagrams are more common, here the diagram edges are just straight line segments.

Definition 2. *A line diagram (also called* embedding [6], Hasse diagram [1] *or simply* diagram [2]*) of a lattice* $\underline{\mathfrak{V}}$ *is a diagram (as previously defined), where*

$$\text{pos}(vw) = \{t \cdot \text{pos}(v) + (1 - t) \cdot \text{pos}(w) \mid \; t \in [0, 1]\}.$$

holds for all elements $v \prec w$.

A lattice is planar if it possesses a plane line diagram, i.e. if no diagram edges intersect [6].

2.2 Lattices and Planarity

In this subsection we give some lattice properties characterizing whether a lattice is planar or not.

Definition 3. *[4] The* (order) dimension $\dim(\underline{P})$ *of an ordered set* $\underline{P}$ *is the smallest cardinal number* m *such that* $\leq$ *is the intersection of* m *linear orders.*

Definition 4. *[4] A* conjugate order L_c *on an ordered set* $\underline{P} = (P, \leq)$ *is a relation meeting the following conditions (*$\|$ *denotes the incomparability relation in* $\underline{P}$*).*

1. L_c *is a strict order*
2. $L_c \cup L_c^{-1} = \|$.

Theorem 1. *[4] Let* $\underline{P} = (P, \leq)$ *be an ordered set. Then the following are equivalent:*

1. $D(\underline{P}) \leq 2$
2. *There exists a conjugate order* L_c *on* $\underline{P}$.

Sketch of proof: Together with L_c, also $R_c := L_c^{-1}$ is a conjugate order. It is easy to show that $L_c \cup \leq$ and $R_c \cup \leq$ are linear orders, the intersection of which is $\leq$. On the other hand, if $K \supseteq \leq$ is a linear order on P then we can show that $K \setminus \leq$ is a conjugate order.