Bialgebraic Contexts for Distributive Lattices - Revisited

Jürg Schmid

Inst. of Math., Univ. of Bern, Bern, Switzerland
juerg.schmid@math-stat.unibe.ch

Dedicated to the memory of my friend Ivan Rival

Abstract. In [8], Vogt used so-called *bialgebraic* contexts to represent the lattice $Sub(L)$ of all sublattices of a finite distributive lattice L as the substructure lattice of an appropriately defined finite (universal) algebra, based on Rival's description (see [4] and [5]) by means of deleting suitable intervals from L. We show how to extend Vogt's context in order to obtain a conceptually simpler description of $Sub_{01}(L)$ - the lattice of all 0-1-preserving sublattices of L - by means of quasiorders and an associated total binary operation on $J(L)^2$, the set of all pairs of non-zero join-irreducibles of L. Our approach is based on Birkhoff- resp. Priestley-duality, a standard reference is [1].

1 Introduction

In the fall of 1974, when we both were visiting at Caltech, Ivan Rival initiated me to the structure theory of distributive lattices by explaining one of his then favourite results: Given a finite distributive lattice L, a nonempty subset $S \subseteq L$ is a sublattice of L iff S can be obtained from L by deleting some collection of intervals $[j, m]$ where j is join-irreducible or 0 and m is meet-irreducible or 1 (see [4] and [5]). Obviously, this description allows us to compute systematically all sublattices of L and thus to determine the substructure lattice of L. As rather small examples already show, the structure of this lattice may be surprisingly complex, and quite a number of papers have been devoted to its study. A good source is [3] and its bibliography.

Rival's characterization of sublattices may be phrased as follows: Membership of an element $x \in L$ in some sublattice of L is equivalent to non-membership of x in some collection of special intervals in L. Writing $\mathcal{D}(L)$ for the collection of all *nonempty* intervals of the type considered, let $\mathbb{K}_L$ be the formal context $(L, \mathcal{D}(L), \notin)$. It follows at once that the extents of $\mathbb{K}_L$ are just the sublattices of L, while the intents are those subsets $B \subseteq \mathcal{D}(L)$ closed under the closure operator given by $\sigma(B) := \{[j, m] \in \mathcal{D}(L);\ [j, m] \subseteq \bigcup B\}$. Consequently, the substructure lattice of L is dually isomorphic to the lattice of all closed sets of the closure system $(\mathcal{D}(L), \sigma)$. It is the purpose of this note to replace this

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description by the lattice of all subalgebras of a (universal) algebra naturally connected with L.

Unless otherwise stated, L will always denote a finite distributive lattice with bounds 0 and 1 (freely confounded with its carrier set), operations $\wedge$ and $\vee$ and order relation $\leq$. We denote by $J(L)$ the set of all join-irreducible elements of L (excluding 0) and by $M(L)$ the set of all meet-irreducibles (excluding 1). For $x \in L$ we write $\downarrow x$ for $\{y \in L,\ y \leq x\}$, and similarly $\uparrow x$ for $\{y \in L,\ y \geq x\}$. Without loss of generality but some gain in the smoothness of presentation, we will only consider 0-1-sublattices of L in section 3: Writing $Sub(L)$ resp. $Sub_{01}(L)$ for the lattice of all sublattices resp. of all 0-1-sublattices of L, we have $Sub(L) \cong Sub_{01}(0' \oplus L \oplus 1')$ with $0' < x < 1'$ for all $x \in L$. A *quasiorder* Q on a set X is a reflexive and transitive binary relation X; given such Q, a subset of $D \subseteq X$ is called a *Q-down-set* iff $x \in X$, $d \in D$ and $(x,d) \in Q$ jointly imply that $x \in D$ (so, in particular, $\downarrow x$ is a $\leq$-down-set in L).

2 Bialgebraic Contexts

In order to represent the subalgebra lattice of a given algebra by that of another (hopefully simpler) algebra, Vogt introduced so-called bialgebraic contexts, implicitly in [7] and explicitly in [8], and used them to describe - in [8] - the lattice $Sub(L)$ of general sublattices of L. We briefly summarize the idea and main results of [8]:

A context $\mathbb{K} = (G, M, I)$ is called *bialgebraic* provided the object set G as well as the attribute set M are carriers of algebras (G, F_G) resp. (M, F_M) such that the extents of $\mathbb{K}$ coincide with the (carriers of) the subalgebras of (G, F_G) and the intents of $\mathbb{K}$ with the (carriers of) the subalgebras of (M, F_M). Note that there is no formal condition linking the types of F_G resp. F_M (although this may be desirable in a concrete application), and that the fundamental operations in of F_G resp. F_M are allowed to be partial.

Looking at the formal context $\mathbb{K}_L$ defined in section 1, we see that it is algebraic on the object half while for the attribute half we have the explicitly given closure operator σ describing the intents. What is missing in order to make $\mathbb{K}_L$ bialgebraic is thus a bunch of (possibly partial) *operations* defined on $\mathcal{D}(L)$ such that the σ-closed subsets of $\mathbb{K}_L$ are exactly those closed under these operations. Finding such operations - in fact, a single partial binary operation will do in the end - takes the major part of [8].

Since the intents of $\mathbb{K}_L$ are exactly the subsets of $\mathcal{D}(L)$ respecting all attribute implications of $\mathbb{K}_L$, Vogt considers the latter as multioperations on $\mathcal{D}(L)$ and extracts from them the sought partial operation(s). The highly nontrivial key step consists in proving that each implication in the canonical minimum Duquenne-Guiges base for (the implications of) $\mathbb{K}_L$ may be simplified in such a way that the resulting set of implications has premises of cardinality at most two and all conclusions are singletons. These conclusions are then considered as the images of their premises under a single partial operation $* : \mathcal{D}(L) \times \mathcal{D}(L) \longrightarrow \mathcal{D}(L)$,