Which Concept Lattices Are Pseudocomplemented?

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Abstract We give a contextual characterization of pseudocomplementation by means of the arrow relations.

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1 Introduction

A lattice L with 0 is *pseudocomplemented* if for each $x \in L$ there is an element $x^* \in L$ (called the *pseudocomplement*[1] of x) such that

$$x \wedge y = 0 \iff y \leq x^*.$$

In this case, $x \mapsto x^*$ defines a unary operation on L, called *pseudocomplementation*, which is automatically antitone and square extensive, i.e., which satisfies

$$x \leq y \Rightarrow y^* \leq x^* \qquad \text{and} \qquad x \leq x^{**}$$

for all $x, y \in L$. These two properties together imply the *join de Morgan law*

$$(x \vee y)^* = x^* \wedge y^*.$$

In fact, we get

Proposition 1. *If L is pseudocomplemented, then*

$$\left(\bigvee X\right)^* = \bigwedge\{x^* \mid x \in X\},$$

whenever $\bigvee X$ exists in L.

Proof. From $x \in X$ we infer $x \leq \bigvee X$ and thus $(\bigvee X)^* \leq x^*$. Therefore $(\bigvee X)^*$ is a lower bound of $\{x^* \mid x \in X\}$. Conversely let y be a lower bound of $\{x^* \mid x \in X\}$, i.e., $y \leq x^*$ for all $x \in X$. Then $y^* \geq x^{**} \geq x$ for all $x \in X$ and thus $y^* \geq \bigvee X$. From this we get $y \leq y^{**} \leq (\bigvee X)^*$. This proves that $(\bigvee X)^*$ is the greatest lower bound of $\{x^* \mid x \in X\}$. □

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[1] The pseudocomplement of x (if it exists) is its greatest semicomplement.

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This proposition offers an easy way to prove pseudocomplementedness: it suffices to exhibit a join-dense set J of elements with pseudocomplement. The pseudocomplement of any other element x is then obtained as the meet of the pseudocomplements of those elements in J which are below x.

For complete lattices there is a simple and rather obvious condition for having a pseudocomplement:

Proposition 2. *Let x be an element of a complete lattice L. Then*

$$x^* \text{ exists} \quad \iff \quad x \wedge \bigvee\{y \in L \mid x \wedge y = 0\} = 0.$$

If x^* exists then

$$x^* = \bigvee\{y \in L \mid x \wedge y = 0\}.$$

Pseudocomplemented lattices, also known as *p-algebras*, have been widely investigated. One of the general sources is the survey by Katriňák [Ka80], but also the books by Balbes and Dwinger [BD74] and Grätzer [Gr71] for the distributive case. Varieties of distributive p-algebras have been described by Lee [Lee70]. The free algebras in these varieties can be used for modeling inconsistent information in databases, see Schmid [Sc88], Sofronie-Stokkermans [So98]. Generalized notions of negation have been studied in [Kw04].

2 Pseudocomplemented Closure Systems

For a concept lattice, being pseudocomplemented is naturally expressed in terms of the closure system of extents. We therefore formulate our observations in the language of closure systems. Let $\mathcal{E}$ be a closure system on a set G, and let

$$A \mapsto A''$$

be the corresponding closure operator. For simplicity we assume $\emptyset'' = \emptyset$ and $g'' = h'' \Rightarrow g = h$; these are merely technical conditions. The closure system $\mathcal{E}$ is called pseudocomplemented if each closed set has a pseudocomplement.

Proposition 3. *A closed set $A \in \mathcal{E}$ has a pseudocomplement A^* if and only if*

$$A \cap \{g \in G \mid A \cap g'' = \emptyset\}'' = \emptyset.$$

In this case,

$$A^* = \{g \in G \mid A \cap g'' = \emptyset\}.$$

If $\{g \in G \mid A \cap g'' = \emptyset\}$ is closed, then it is the pseudocomplement of A.

Proof. The first claim is the same as in Proposition 2 with A in the rôle of x, except that we have simplified the right hand side:

$$\begin{aligned}\bigvee\{B \in \mathcal{E} \mid A \cap B = \emptyset\} &= \left(\bigcup\{B \in \mathcal{E} \mid A \cap B = \emptyset\}\right)'' \\ &= \{g \in G \mid A \cap g'' = \emptyset\}'',\end{aligned}$$