This chapter grounds lattice theory

Mathematical theories abound; however, many of them are not currently useful. On the other hand, popular practical techniques are often short of a sound mathematical basis. Hence, the latter techniques are short of instruments to optimize their application. It turns out that lattice theory is firmly grounded in the real world. This chapter illustrates how.

### 4.1 The Euclidean Space $\mathbb{R}^N$

The Euclidean space $\mathbb{R}^N$ is the Cartesian product $\mathbb{R}^N = \mathbb{R} \times \cdots \times \mathbb{R}$, where the set $\mathbb{R}$ of real numbers has emerged from the conventional measurement process of successive comparisons (Goldfarb and Deshpande 1997). People have used (and studied) real numbers for millennia, and essential properties of the set $\mathbb{R}$ are known. For instance, $\mathbb{R}$ is a (mathematical) field; moreover, there is a metric in $\mathbb{R}$; furthermore, the cardinality $\aleph_1$ of $\mathbb{R}$ is larger than the cardinality $\aleph_0$ of the subset $\mathbb{Q}$ of rational numbers – For definitions of a mathematical field, metric, etc. see in Appendix A.

Practical decision-making is often based on the total ordering property of the non-complete lattice $\mathbb{R}$. The meet ($\wedge$) and join ($\vee$) in the chain $\mathbb{R}$ are given, respectively, by $x\wedge y = \min\{x, y\}$ and $x\vee y = \max\{x, y\}$.

Mathematical studies were extended from $\mathbb{R}$ to $\mathbb{R}^N$. The linear space $\mathbb{R}^N = \mathbb{R} \times \cdots \times \mathbb{R}$ emerged ‘par excellence’ as the modeling domain in various applications. However, the product lattice $\mathbb{R}^N = \mathbb{R} \times \cdots \times \mathbb{R}$ is not totally ordered; rather, $\mathbb{R}^N$ is partially ordered, as illustrated in Fig. 4-1 on the plane for $N=2$, with partial ordering $(x_1, \ldots, x_N) \leq (y_1, \ldots, y_N) \iff x_1 \leq y_1, \ldots, x_N \leq y_N$. An interval in lattice $\mathbb{R}^N$ is a N-dimensional hyperbox, or hyperbox for short. The set of hyperboxes in $\mathbb{R}^N$ is a non-complete lattice.

A strictly increasing real function on $\mathbb{R}$ is a positive valuation function as shown in section 4.5, where a positive valuation is also defined in a lattice of intervals. Popular in applications is the ‘unit hypercube’ as explained in the following.


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4.2 Hyperboxes in $\mathbb{R}^N$

A complete lattice, denoted by $U$, is defined in the $N$-dimensional unit hypercube with partial ordering $(x_1,\ldots,x_N) \leq (y_1,\ldots,y_N) \iff x_1 \leq y_1, \ldots, x_N \leq y_N$. In the interest of simplicity the dimension (N) was suppressed in symbol $U$. Note that consideration of the lattice $U$, instead of $\mathbb{R}^N$, is not restrictive by default, since all physical quantities have upper/lower bounds and suitable transformations to the unit hypercube can be found. Our interest is in learning sets of points using a finite collection of hyperboxes.

Lattice $U$ is the product of $N$ identical constituent lattices, these are the complete lattices (chains) $I=[0,1]$ each with least element $0=0$ and greatest element $1=1$. It follows that $U$ is a complete lattice with least element $(0,\ldots,0)$ and greatest element $(1,\ldots,1)$. Moreover, $\tau(U)$ is an atomic lattice because every hyperbox is the lattice-join of atoms ‘min’ and ‘MAX’.

A number of computational intelligence schemes that learn by computing hyperboxes have been proposed including min-max neural networks (Simpson 1992, 1993; Gabrys and Bargiela 2000) as well as adaptive resonance theory (ART) inspired neural networks (Georgiopoulos et al. 1994; Healy and Caudell 1997; Cano Izquierdo et al. 2001; Anagnostopoulos and Georgiopoulos 2002; Parrado-Hernández et al. 2003; Castro et al. 2005). Moreover, in a machine-learning context, the class of axis-parallel rectangles was shown to be efficiently probably approximately correct (PAC) learnable (Blumer et al. 1989; Kearns and Vazirani 1994; Long and Tan 1998); other applications have also considered hyperboxes for learning (Salzberg 1991; Wettschereck and Dietterich 1995; Dietterich et al. 1997).