11. Distributions and Green’s Functions

Whoever understands Green’s functions can understand forces in nature. 
Folklore

The invention of the Green’s function brought about a tool-driven revolution in mathematical physics, similar in character to the more famous tool-driven revolution caused by the invention of electronic computers a century and a half later… The Green’s function and the computer are prime examples of intellectual tools. They are tools for clear thinking…

Invented in 1828 by George Green (1793–1841) and successfully applied to classical electromagnetism, acoustics, and hydrodynamics, Green’s functions were the essential link between the theories of quantum electrodynamics by Schwinger, Feynman, and Tomonaga in 1948 and are still alive and well today…

I began the application of the Green’s function to condensed matter physics in 1956 with a study of spin-waves in ferromagnets. I found that all the Green’s function tricks that had worked so well in quantum electrodynamics worked even better in the theory of spin waves…

Meanwhile, the Green’s functions method was applied systematically by Bogoliubov and other people to a whole range of problems in condensed matter physics. The main novelty in condensed matter physics was the appearance of temperature as an additional variable… A beautiful thing happens when you make the transition from ordinary Green’s functions to thermal Green’s functions. To make the transition, all you have to do is to replace the real frequency of any oscillation by a complex number whose real part is frequency and whose imaginary part is temperature¹…

Soon after thermal Green’s functions were invented, they were applied to solve the outstanding unsolved problem of condensed matter physics, the problem of superconductivity. They allowed Cooper, Bardeen, and Schrieffer to understand superconductivity as an effect of a particular thermal Green’s functions expressing long-range phase-coherence between pairs of electrons (called Cooper pairs)…

In the 1960s, after Green’s functions had become established as the standard working tools of theoretical analysis in condensed matter physics, the wheel of fashion in particle physics continued to turn. For a decade, quantum field theory and Green’s functions were unfashionable in particle

physics. The prevailing view was that quantum field theory had failed in the domain of strong interactions\(^2\) ...

Then in the 1970s, the wheel of fashion turned once more. Quantum field theory was back in the limelight with two enormous successes, the Weinberg–Salam unified theory of electromagnetic and weak interactions, and the gauge theory of strong interactions now known as quantum chromodynamics. Green’s functions were once again the working tools of calculation, both in particle physics and in condensed matter physics. And so they have remained up to the present day.

In the 1980’s, quantum field theory moved off in a new direction, to lattice gauge theories in one direction and to superstring theory in another... The Wilson loop is the reincarnation of a Green’s function in a lattice gauge theory\(^3\) and there is a reincarnation of Green’s functions in superstring theory.

Freeman Dyson

*George Green and physics*\(^4\)

Between 1930 and 1940, several mathematicians began to investigate systematically the concept of a “weak” solution of a linear partial differential equation, which appeared episodically (and without a name) in Poincaré’s work.

It was one of the main contributions of Laurent Schwartz when he saw, in 1945, that the concept of distribution introduced by Sobolev in 1936 (which he had rediscovered independently) could give a satisfactory generalization of the Fourier transform including all the preceding ones... By his own research and those of his numerous students, Laurent Schwartz began to explore the potentialities of distributions (generalized functions) and gradually succeeded in convincing the world of mathematicians that this new concept should become central in all problems of mathematical analysis, due to the greater freedom and generality it allowed in the fundamental operations of calculus, doing away with a great many unnecessary restrictions and pathology.

The role of Laurent Schwartz (born 1915) in the theory of distributions is very similar to the one played by Newton (1643–1727) and Leibniz (1646–1716) in the history of Calculus. Contrary to popular belief, they of course did not invent it, for derivation and integration were practiced by men such as Cavalieri (1598–1647), Fermat (1601–1665) and Roberval (1602–1675) when Newton and Leibniz were merely schoolboys. But they were able to systematize the algorithms and notations of Calculus in such a way that it became a versatile and powerful tool which we know, whereas before them it could only be handled via complicated arguments and diagrams.

Jean Dieudonné, 1981

*History of Functional Analysis*\(^5\)

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\(^4\) Physics World, August 1993, pp. 33–38 (reprinted with permission).

\(^5\) North Holland, Amsterdam, 1981 (reprinted with permission).