Random matrices,
non-colliding processes and queues

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This is a survey of some recent results connecting random matrices, non-colliding processes and queues.

1 Introduction

It was recently discovered by Baik, Deift and Johansson [4] that the asymptotic distribution of the length of the longest increasing subsequence in a permutation chosen uniformly at random from $S_n$, properly centred and normalised, is the same as the asymptotic distribution of the largest eigenvalue of an $n \times n$ GUE random matrix, properly centred and normalised, as $n \to \infty$. This distribution had earlier been identified by Tracy and Widom [54] in the random matrix context, and it is now known as the Tracy-Widom law.

The length of the longest increasing subsequence in a random permutation can be thought of as a 'last-passage time' for a certain directed percolation problem on the plane; this directed percolation problem is closely related to the one-dimensional totally asymmetric exclusion process (with low density of particles travelling at high speed) or equivalently, an infinite series of $M/M/1$ queues in tandem (with low density of customers and high service rates). On the other hand, the eigenvalues of a GUE random matrix of dimension $n$ have the same law as the positions, after a unit length of time, of $n$ independent standard Brownian motions started from the origin and conditioned (in the sense of Doob) never to collide. These remarks are for the purpose of convincing the reader that there might be some interesting connections between random matrices, non-colliding processes and queues. Indeed there are, and that is the topic of this paper.

Let us concentrate on the following, more exact, connection between directed percolation and random matrices which was more recently observed by Baryshnikov [5] and Gravner, Tracy and Widom [26]. Let $B = (B_1, \ldots, B_n)$ be a standard $n$-dimensional Brownian motion, and write $B(s, t) = B(t) - B(s)$ for $s < t$.

**Theorem 1** [Baryshnikov; Gravner-Tracy-Widom] The random variable

$$M_n = \sup_{0 \leq s_1 \leq \cdots \leq s_{n-1} \leq 1} \{B_n(0, s_1) + \cdots + B_1(s_{n-1}, 1)\}$$

has the same law as the largest eigenvalue of an $n$-dimensional GUE random matrix.
The proofs of this fact given in [5] and [26] are based on the Robinson-Shensted-Knuth (RSK) correspondence, and do not make use of the facts that

(a) $M_n$ has a queueing interpretation [24], and
(b) the largest eigenvalue of a GUE random matrix has an interpretation in terms of non-colliding Brownian motions [21, 25].

In [47], a proof is given of a more general result which is based entirely on these interpretations. This more general result identifies a path-transformation $\Gamma_n(B)$ of $B$, which has the same law as that of $n$ independent Brownian motions conditioned (in the sense of Doob) never to collide. This process, which we denote by $\hat{B}$, is the eigenvalue process associated with Hermitian Brownian motion and $\hat{B}(1)$ has the same distribution as the eigenvalues of an $n$-dimensional GUE random matrix [21, 25]. The transformed process $\Gamma_n(B)$ has the property that its largest component at time 1 is given by $M_n$, and so Theorem 1 follows.

At the heart of the proof, which will be outlined in this paper, is a celebrated theorem of classical queueing theory, which states that, in equilibrium, the output of a stable $M/M/1$ queue is Poisson. This result is usually attributed to Burke, who gave the first proof in 1956. In what follows, we shall also refer to it as an ‘output theorem’. It follows from the reversibility of the $M/M/1$ queue. There is an easy extension of this theorem, which can be proved by similar reversibility arguments; when phrased in the setting of ‘max-plus algebra’, this extension immediately yields the two-dimensional result. The result in higher dimensions is then obtained by considering a series of queues in tandem and applying an induction argument. The statement of Theorem 1 seems considerably less mysterious (to me at least!) in the setting of queues and non-colliding processes.

In the case $n = 2$, the fact that $\Gamma_n(B)$ and $\hat{B}$ have the same law is equivalent to Pitman’s representation for the three-dimensional Bessel process. In the case $n = 3$, it yields a representation for planar Brownian motion conditioned to stay forever in a wedge of angle $\pi/3$; a partial converse of this result was discovered earlier by Biane [6].

As we remarked above, the process $\hat{B}$ has the same law as the eigenvalue process associated with Hermitian Brownian motion. Bougerol and Jeulin [8] have (independently) obtained a similar representation for this process, by completely different methods, which is also consistent with Theorem 1. Their results are presented in the more general setting of Brownian motion on symmetric spaces. The relationship between these two representations will be discussed elsewhere.

There are certain symmetries in the max-plus algebra which play a crucial role, and these symmetries also exist in other algebras, including the conventional algebra on the reals. As a consequence, there are analogues of these output theorems in many different settings, and it seems that there are some general phenomena at work. In all cases, symmetry and reversibility play a key role. We will present some of these examples and their implications.

We will also describe briefly how output theorems and the study of tandem systems can be used to obtain first order asymptotic results for various directed percolation