Gaussian maximum of entropy and reversed log-Sobolev inequality

Djalil Chafai  chafai@cict.fr

Laboratoire de Statistique et Probabilités, UMR CNRS 55830
Université Paul Sabatier, F-31062 Cédex, Toulouse, France.

Abstract. The aim of this note is to connect a reversed form of the Gross logarithmic Sobolev inequality with the Gaussian maximum of Shannon's entropy power. There is thus a complete parallel with the well-known link between logarithmic Sobolev inequalities and their information theoretic counterparts. We moreover provide an elementary proof of the reversed Gross inequality via a two-point inequality and the Central Limit Theorem.

1 Shannon's entropy power and Gross' inequality

In the sequel, we denote by $\text{Ent}_\mu(f)$ the entropy of a non-negative integrable function $f$ with respect to a positive measure $\mu$, defined by

$$\text{Ent}_\mu(f) := \int f \log f \, d\mu - \int f \, d\mu \log \int d\mu.$$ 

The Shannon entropy [15] of an $n$-variate random vector $X$ with probability density function (pdf) $f$ is given by

$$H(X) := -\text{Ent}_{\lambda_n}(f) = -\int f \log f \, dx,$$

where both $\lambda_n$ and $dx$ denote the $n$-dimensional Lebesgue measure on $\mathbb{R}^n$. The Shannon entropy power [15] of $X$ is then given by

$$N(X) := \frac{1}{2\pi e} \exp \left( \frac{2}{n} H(X) \right).$$

It is well-known (cf. [15,8]) that Gaussian saturates this entropy at fixed covariance. Namely, for any $n$-variate random vector $X$ with covariance matrix $K(X)$, one have

$$N(X) \leq |K(X)|^{1/n},$$

and $|K|^{1/n}$ is the entropy power of the $n$-dimensional Gaussian with covariance $K$.

The logarithmic Sobolev inequality of Gross [11] expresses that for any non-negative smooth function $f : \mathbb{R}^n \to \mathbb{R}^+$

$$2 \text{Ent}_{\gamma_n}(f) \leq E_{\gamma_n} \left( \frac{\nabla f^2}{f} \right),$$

(2)

© Springer-Verlag Berlin Heidelberg 2003
where $E_{\gamma_n}$ denotes the expectation with respect to $\gamma_n$, $|\cdot|$ the Euclidean norm and $\gamma_n$ the $n$-dimensional standard Gaussian given by

$$d\gamma_n(x) := (2\pi)^{-\frac{n}{2}} e^{-\frac{|x|^2}{2}} dx.$$  

Inequality (2) is sharp and the equality is achieved for $f$ of the form $\exp(a \cdot )$.

By performing a change of function and an optimization, Becker showed [4] (see also [7]) that (2) is equivalent to the following "Euclidean" logarithmic Sobolev inequality, for any pdf $g$

$$\text{Ent}_{\lambda_n}(g) \leq \frac{n}{2} \log \left[ \frac{1}{2\pi n} \int \frac{\nabla |g|^2}{g} dx \right],$$  

(3)

where $\lambda_n$ is the $n$-dimensional Lebesgue measure on $\mathbb{R}^n$. Therefore, for any $n$-variate random vector $X$ (with pdf $g$), we have

$$N(X)J(X) \geq n.$$  

(4)

This inequality can be obtained by many methods. The most classical ones are via Shannon’s entropy power inequality together with DeBruijn identity, or via Stam’s super-additivity of the Fisher information (cf. [16,7,10,1]). Moreover, Dembo showed in [9] that (4) is equivalent to

$$N(X)|J(X)|^{1/n} \geq 1,$$  

(5)

where $J(X)$ is the Fisher information matrix of $X$ defined by

$$J(X) := \int \nabla \log g \cdot \nabla \log g^T g \, dx,$$

and we have $J(X) = \text{Tr} J(X)$. To deduce (5) from (4), apply (4) to the random vector $X = K(Y)^{-1/2} Y$. Conversely, use the arithmetic-geometric means inequality

$$(a_1 \cdots a_n)^{\frac{1}{n}} \leq \frac{a + \cdots + a_n}{n}$$  

(6)

on the spectrum of the non-negative symmetric matrix $J(X)$.

2 Reversed Gross’ logarithmic Sobolev inequality

The Gross logarithmic Sobolev inequality (2) admits a reversed form which states that for any positive smooth function $f: \mathbb{R}^n \to \mathbb{R}^+$

$$\frac{\|E_{\gamma_n}(\nabla f)^2\|^2}{E_{\gamma_n}(f)} \leq 2 \text{Ent}_{\gamma_n}(f).$$  

(7)