3 Infinitesimal deformations

Let \((X, 0) \subset (\mathbb{C}^N, 0)\) be an analytic germ, with local ring \(\mathcal{O}_{X,0}\). In this section we always consider germs at the origin, and therefore we drop the 0 from the notation. The ring \(\mathbb{C}[x]\) of power series in \(N\) variables will be denoted by \(\mathcal{O}_N\). The first few terms of the resolution of \(\mathcal{O}_X\) are

\[
0 \leftarrow \mathcal{O}_X \leftarrow \mathcal{O}_N \xleftarrow{f} \mathcal{O}_N^k \xleftarrow{r} \mathcal{O}_N^l.
\]

The entries of the row vector \(f = (f_1, \ldots, f_k)\) generate the ideal \(I\) of \(X\), and the columns of \(r\) generate the relations.

A first order infinitesimal deformation of \(X\) is a deformation over the double point \(\mathbb{D}\), the zero-dimensional space with as local ring the ring of dual numbers \(\mathbb{C}[t]/(t^2)\). We also write this ring as \(\mathbb{C}[\varepsilon]\), where \(\varepsilon\) is defined as variable with the property that \(\varepsilon^2 = 0\). So let \(X \to \mathbb{D}\) be a deformation of \(X\). Then there is a resolution

\[
0 \leftarrow \mathcal{O}_X \leftarrow \mathcal{O}_{\mathbb{C}^N \times \mathbb{D}} \xleftarrow{F} \mathcal{O}_{\mathbb{C}^N \times \mathbb{D}}^k \xleftarrow{R} \mathcal{O}_{\mathbb{C}^N \times \mathbb{D}}^l.
\]

with \(F = f + \varepsilon f'\) and \(R = r + \varepsilon r'\). As \(\varepsilon^2 = 0\), the condition \(FR = 0\) gives

\[
FR = (f + \varepsilon f')(r + \varepsilon r') = fr + \varepsilon(f r' + f' r) = 0.
\]

Because \(fr = 0\), we obtain the equation \(f r' + f' r = 0\) in \(\mathcal{O}_N\).

The first order deformations form an \(\mathcal{O}_X\)-module: if \((f + \varepsilon f_1')(r + \varepsilon r_1') = 0\) and \((f + \varepsilon f_2')(r + \varepsilon r_2') = 0\), then \((f + \varepsilon (f_1' + f_2'))(r + \varepsilon (r_1' + r_2')) = (fr + \varepsilon(f(r_1' + r_2') + (f_1' + f_2') r)) = 0\). Furthermore, \((f + \varepsilon \phi f')(r + \varepsilon \phi r') = 0\) for \(\phi \in \mathcal{O}_N\). Finally, if \(f' \in \mathcal{O}_N^k \subset \mathcal{O}_N^k\), then there exists a matrix \(M \in M_k(\mathcal{O}_N)\) with \(f + \varepsilon f' = f(1 + \varepsilon M)\); as \(1 + \varepsilon M\) is invertible (with inverse \(1 - \varepsilon M\)), the ideals generated by \(f\) and by \(f + \varepsilon f'\) are equal.

**Proposition.** The \(\mathcal{O}_X\)-module of first order deformations is isomorphic to the normal module \(N_X = \text{Hom}_{\mathcal{O}_X}(I/I^2, \mathcal{O}_X)\).

**Proof.** Let \(f + \varepsilon f'\) be an infinitesimal deformation. Then \(f'\) determines an \(\mathcal{O}_N\)-homomorphism \(\mathcal{O}_N^k \to \mathcal{O}_N\), which maps the image of \(r\) into \(I\), because \(f' r = -f r\). Hence \(f'\) induces a homomorphism \(\rho(f') : \mathcal{O}_N^k / \text{Im} r = I \to \mathcal{O}_N / I = \mathcal{O}_X\), which sends \(f_i\) to \((f_i' \mod I)\). This means that \(\rho(f')\) is an element of \(\text{Hom}_{\mathcal{O}_X}(I, \mathcal{O}_X) \cong \text{Hom}_{\mathcal{O}_X}(I/I^2, \mathcal{O}_X)\).
3 Infinitesimal deformations

Conversely, given a homomorphism \( \varphi \in \text{Hom}_\mathcal{O}_X(I, \mathcal{O}_X) \), we lift the vector
\[
\varphi(f) = (\varphi(f_1), \ldots, \varphi(f_k)) \in \mathcal{O}_X^k
\]
to a vector \( f' \in \mathcal{O}_N^k \), inducing a homomorphism \( \tilde{\varphi}: \mathcal{O}_N^k \to \mathcal{O}_N \). For every relation \( r_j \) the function \( f'r_j = \tilde{\varphi}(r_j) \) is a lift of \( \varphi(\sum f_i r_{ij}) = 0 \in \mathcal{O}_X \). Therefore one can find a matrix \( r' \) with \( f'r + f' = 0 \). Any two liftings of \( \varphi(f) \) differ by a \( g \in I^k \), so they determine the same deformation. \( \square \)

An infinitesimal deformation \( f + \varepsilon f' \) is trivial, if there is an automorphism \( \varphi(x, \varepsilon) = (x + \varepsilon \delta(x), \varepsilon) \) of \( \mathbb{C}^N \times \mathbb{D} \), such that \( f + \varepsilon f' \) and \( f \circ \varphi \) determine the same ideal. Let \( \mathcal{O}_X \) be module of germs of vector fields at the origin. The computation
\[
\frac{d}{d\varepsilon} f \circ \varphi(x, \varepsilon) \big|_{\varepsilon=0} = \frac{d}{d\varepsilon} f(x + \varepsilon \delta(x)) \big|_{\varepsilon=0} = \sum_j \frac{\partial f}{\partial x_j} \delta_j(x).
\]
shows that the trivial deformations are the image of the natural map
\[
\mathcal{O}_N|_X = \mathcal{O}_N \otimes \mathcal{O}_X \to \text{Hom}_\mathcal{O}_X(I, \mathcal{O}_X) = N_X.
\]
which sends a vector field \( \delta \) to the homomorphism \( g \mapsto \delta(g) \). The kernel of this map is the \( \mathcal{O}_X \)-module \( \mathcal{O}_X = \{ \delta|_X \mid \delta(I) \subseteq I \} \). One has \( \mathcal{O}_X = \text{Hom}_\mathcal{O}_X(\mathcal{O}_I^1, \mathcal{O}_X) \).

**Definition.** The module \( T^1_X \) of isomorphism classes of first order infinitesimal deformations is
\[
T^1_X = \text{coker}\{ \mathcal{O}_N|_X \to N_X \}.
\]

**Example: hypersurface singularities.** In this case the ideal \( I \) is principal, generated by a function \( f \), and \( N_X = \text{Hom}(I/I^2, \mathcal{O}_X) \) is a free \( \mathcal{O}_X \)-module with \( f \) as generator. Therefore
\[
T^1_X = \mathcal{O}_{n+1}/(f, \frac{\delta f}{\partial x_1}, \ldots, \frac{\delta f}{\partial x_n}).
\]

**Remark.** If the germ \( X \) is smooth, then \( T^1_X = 0 \). In fact, we may assume that we have equality of germs \( (\mathbb{C}^N, 0) = (X, 0) \), so \( N_X = 0 \). In the algebraic situation, for affine \( X \), this argument does not work, but it is well known that for a nonsingular subvariety \( X \) of a nonsingular variety \( Y \) the sequence of sheaves
\[
0 \to \mathcal{O}_X \to \mathcal{O}_Y \otimes \mathcal{O}_X \to N_X \to 0
\]
is exact [Ha, p. 182].

Until now we worked with modules over the local ring at the origin. Our definitions extend to give a coherent sheaf \( T^1_X \) on an analytic space \( X \). If the germ \( (X, 0) \) has an isolated singularity, then by coherence of this sheaf \( T^1_X \) is a finite-dimensional vector space.