Chapter VIII. Simplicial sets and topological spaces

§1. Introduction

The purpose of this chapter is

(i) to review some of the basic notions of simplicial homotopy theory, and

(ii) to convince (or at least try to convince) the reader that this simplicial homotopy theory is equivalent to the usual topological homotopy theory.

In slightly more detail:

§2. Here we define simplicial sets, give a few examples and construct the singular and realization functors between the category of simplicial sets and the category of topological spaces.

§3 contains Quillen's precise formulation of the sense in which the singular and realization functors induce an "equivalence between the homotopy theories of the categories and ". For this one needs in both categories notions of fibrations, cofibrations and weak equivalences.

§4. We end the chapter with a discussion of the homotopy relation for simplicial maps and review the related notion of function spaces for simplicial sets.

For a more detailed account of simplicial homotopy theory the reader may consult [May], [Lamotke], [Curtis (S)], [Gugenheim], [Quillen (HA)] and others.
§2. Simplicial sets

In this section we

(i) recall a definition of simplicial sets and, more generally, of simplicial objects over an arbitrary category,

(ii) discuss some simple examples of simplicial sets, and

(iii) observe that the categories $\mathcal{S}$ of simplicial sets and $\mathcal{T}$ of topological spaces are related by a pair of adjoint functors

$$\mathcal{S} \xrightarrow{\text{Realization}} \mathcal{T} \xleftarrow{\text{Singular}}$$

the realization functor $\mid : \mathcal{S} \to \mathcal{T}$ and the singular functor $\text{Sin} : \mathcal{T} \to \mathcal{S}$.

We start with

2.1 Simplicial objects and maps. A simplicial object $X$ over a category $\mathcal{C}$ consists of

(i) for every integer $n \geq 0$ an object $X_n \in \mathcal{C}$, and

(ii) for every pair of integers $(i,n)$ with $0 \leq i \leq n$, face and degeneracy maps

$$d_i : X_n \to X_{n-1} \quad \text{and} \quad s_i : X_n \to X_{n+1}$$

satisfying the simplicial identities: