III. LANGLANDS'S CONSTRUCTION OF THE TANIYAMA GROUP

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Introduction

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Introduction: In this article we give a detailed description of Langlands's construction of his Taniyama group. The first section reviews the definition and properties of the Serre group, and the following section discusses extensions of Galois groups by the Serre group. The construction itself is carried out in the third section, which also contains additional material required for V.

We mention that in [1] Langlands is using the opposite sign convention for the reciprocity law in class field theory from us and hence the opposite notion of the Weil group (although his statement at the bottom of p. 224 is misleading on this point). Thus, there are many sign differences between his article and ours.

Notation: Vector spaces are finite-dimensional, number fields are of finite degree over $\mathbb{Q}$ (and usually contained in $\overline{\mathbb{Q}}$), and $\overline{\mathbb{Q}}$ is the algebraic closure of $\mathbb{Q}$ in $\mathbb{C}$. For $L$ a number field, $L^\text{ab} \subset \overline{\mathbb{Q}}$ denotes its abelian closure. For the Weil group, we follow the notations of Tate [2]. In particular, for a topological group $\Gamma$, $\Gamma^C$ denotes the closure of the commutator subgroup of $\Gamma$ and $\Gamma^\text{ab} = \Gamma/\Gamma^C$. 
§1. The Serre group.

Let \( L \subset \mathbb{C} \) be a finite extension of \( \mathbb{Q} \), let \( \Gamma \) be the set of embeddings of \( L \) into \( \mathbb{C} \), and write \( L^x \) for \( \text{Res}_{L/\mathbb{Q}} \mathbb{C} \). Any \( \rho \in \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \) defines an element \([\rho]\) of \( \Gamma \), which may be regarded as a character of \( L^x \). Then \( \Gamma \) is a basis for \( X^*(L^x) \). An element \( \sigma \) of \( \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \) acts on \( X^*(L^x) \) by
\[
\sigma(\Sigma b_\rho [\rho]) = \Sigma b_\rho [\sigma \rho] = \Sigma b_{\sigma^{-1} \rho} [\rho].
\]
The quotient of \( L^x \) by the Zariski closure of any sufficiently small arithmetic subgroup has character group \( X^*(L^x) \cap (Y^O \oplus Y^-) \) where
\[
Y^O = \{ \chi \in X^*(L^x) \oplus \mathbb{Q} | \sigma \chi = \chi, \text{ all } \sigma \in \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \}
\]
\[
Y^- = \{ \chi \in X^*(L^x) \oplus \mathbb{Q} | c \chi = -\chi, \text{ all } c \text{ of the form } c = \sigma \sigma^{-1} \}
\]
(Serre [1, II-31, Cor.1]). Thus this quotient is independent of the arithmetic subgroup; it is called the Serre group \( S^L \) of \( L \) (or, sometimes, the connected Serre group). One checks easily that \( X^*(S^L) \) is the subgroup of \( X^*(L^x) \) of \( \chi \) satisfying
\[
(\sigma - 1)(\sigma + 1) \chi = 0 = (\sigma + 1)(\sigma - 1) \chi, \text{ all } \sigma \in \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}).
\]
There is a canonical homomorphism \( h = h^L : S \to S^L \) and hence corresponding homomorphisms \( w_h : \mathbb{G}_m \to S^L \) and \( \mu = \mu^L : \mathbb{G}_m \to S^L \). They determine the following maps on the character groups:
\[
X^*(h) = (\Sigma b_\rho [\rho] \mapsto (b_1, b_1) : X^*(S^L) \to X^*(S) = \mathbb{Z} \oplus \mathbb{Z})
\]
\[
X^*(w_h) = (\Sigma b_\rho [\rho] \mapsto -b_1 - b_1)
\]
\[
X^*(\mu) = (\Sigma b_\rho [\rho] \mapsto b_1)
\]