Disjoint Unit Spheres admit at Most Two Line Transversals

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Abstract. We show that a set of n disjoint unit spheres in $\mathbb{R}^d$ admits at most *two* distinct geometric permutations, or line transversals, if n is large enough. This bound is optimal.

1 Introduction

A line ℓ is a *line transversal* for a set $\mathcal{S}$ of pairwise disjoint convex bodies in $\mathbb{R}^d$ if it intersects every element of $\mathcal{S}$. A line transversal defines two linear orders on $\mathcal{S}$, namely the order in which ℓ intersects the bodies, where we can choose to orient ℓ in two directions. Since the two orders are essentially the same (one is the reverse of the other), we consider them as a single *geometric permutation.*

Bounds on the maximum number of geometric permutations were established about a decade ago: a tight bound of $2n - 2$ is known for $d = 2$ [2], for higher dimension the number is in $\Omega(n^{d-1})$ [6] and in $O(n^{2d-2})$ [10]. The gap was closed for the special case of spheres by Smorodinsky et al. [9], who showed that n spheres in $\mathbb{R}^d$ admit $\Theta(n^{d-1})$ geometric permutations. This result can be generalized to "fat" convex objects [8].

The even more specialized case of congruent spheres was treated by Smorodinsky et al. [9] and independently by Asinowski [1]. They proved that n unit circles in $\mathbb{R}^2$ admit at most two geometric permutations if n is large enough (the proof by Asinowski holds for all $n \geq 4$). Zhou and Suri established an upper bound of 16 for all d and n sufficiently large, a result quickly improved by Katchalski, Suri, and Zhou [7] and independently by Huang, Xu, and Chen [5] to 4. When the spheres are not congruent, but the ratio of the radii of the largest and smallest sphere is bounded by γ, then the number of geometric permutations is bounded by $O(\gamma^{\log \gamma})$ [12].

Katchalski et al. show that for n large enough, two line transversals can make an angle of at most $O(1/n)$ with each other, so all line transversals are "essentially" parallel. They define a *switched pair* to be a pair of spheres (A, B) such that there are two line transversals ℓ and ℓ' (for all n spheres) where ℓ visits A before B, while ℓ' visits B before A. Katchalski et al. prove that any sphere

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can participate in at most one switched pair, and that the two spheres forming a switched pair must appear consecutively in any geometric permutation of the set. It follows that any two geometric permutations differ only in that the elements of some switched pair may have been exchanged. Katchalski et al.'s main result is that there are at most two switched pairs in a set of n disjoint unit spheres, implying the bound of four geometric permutations.

We show that in fact there cannot be more than one switched pair. This implies that, for n large enough, a set of n disjoint unit spheres admits at most two geometric permutations, which differ only by the swapping of two adjacent elements. Since there are arbitrarily large sets of unit spheres in $\mathbb{R}^d$ with one switched pair, this bound is optimal.

Surveys of geometric transversal theory are Goodman et al. [3] and Wenger [11]. The latter also discusses Helly-type theorems for line transversals. A recent result in that area by Holmsen et al. [4] proves the existance of a number $n_0 \leq 46$ such that the following holds: Let $\mathcal{S}$ be a set of disjoint unit spheres in $\mathbb{R}^3$. If every n_0 members of $\mathcal{S}$ have a line transversal, then $\mathcal{S}$ has a line transversal. Our present results slightly simplify the proof of this result.

2 The Proof

A *unit sphere* is a sphere of radius 1. We say that two unit spheres are *disjoint* if their interiors are (in other words, we allow the spheres to touch). A line *stabs* a sphere if it intersects the closed sphere (and so a tangent to a sphere stabs it). A *line transversal* for a set of disjoint unit spheres is a line that stabs all the spheres, with the restriction that it is not allowed to be tangent to two spheres in a common point (as such a line does not define a geometric permutation).

Given two disjoint unit spheres A and B, let $g(A, B)$ be their center of gravity and $\Pi(A, B)$ be their bisecting hyperplane. If the centers of A and B are a and b, then $g(A, B)$ is the mid-point of a and b, and $\Pi(A, B)$ is the hyperplane through $g(A, B)$ orthogonal to the line ab.

We first repeat a basic lemma by Katchalski et al.

Lemma 1. [7, Lemma 2.3] *Let ℓ and ℓ' be two different line transversals of a set $\mathcal{S}$ of n disjoint unit spheres in $\mathbb{R}^d$. Then the angle between the direction vectors of ℓ and ℓ' is $O(1/n)$.*

Proof. A volume argument shows that the distance between the first and last sphere stabbed by ℓ is $\Omega(n)$. Since ℓ and ℓ' have distance at most 2 over an interval of length $\Omega(n)$, their direction vectors make an angle of $O(1/n)$.

Lemma 1 implies that all line transversals for a set of spheres are nearly parallel.

We continue with a warm-up lemma in two dimensions.

Lemma 2. *Let S and T be two unit-radius disks in $\mathbb{R}^2$ with centers $(-\lambda, 0)$ and $(\lambda, 0)$, where $\lambda \geq \cos\beta$ for some angle β with $0 < \beta \leq \pi/2$. Then $S \cap T$ is contained in the ellipse*

$$\left(\frac{x}{\sin^2\beta}\right)^2 + \left(\frac{y}{\sin\beta}\right)^2 \leq 1.$$