Chapter 5 described how to determine all generators and relations of the cohomology ring out to degree $N$. Carlson presented in [18] a criterion which allows one to conclude that there are no further generators or new relations in higher degrees. In the current chapter we shall recall Carlson’s criterion and discuss how to put it into practice.

One part of the criterion is a condition on the Koszul complex, so the definition of this complex is recalled in Sect. 6.1. Then Carlson’s criterion is stated in Sect. 6.2. Section 6.3 starts by recalling a few facts about regular sequences and then proceeds to explain how the package Diag chooses a system of parameters for the cohomology ring and calculates the depth.

If the group $G$ has small $p$-rank then the cohomology of the Koszul complex and the Poincaré series of the cohomology ring are easier to determine. This is the business of Sect. 6.4, whose results were definitely known to the experts.

Carlson’s criterion allows one to compute the cohomology rings of all groups of order $p^n$ by induction in $n$. Section 6.5 describes how the package Diag handles cohomology rings of subgroups in order to ensure that the cohomology ring of an isomorphism class of $p$-groups is only calculated once.

6.1 The Koszul complex

Koszul complexes play an important role in Carlson’s completeness criterion. As Carlson explains in [18], it makes sense here to view them as cochain complexes.

**Hypothesis 6.1.** Let $A = \bigoplus_{n \geq 0} A^n$ be a graded commutative $k$-algebra with $A^0 = k$, which is generated by finitely many homogeneous elements of $A^+ := \bigoplus_{n > 0} A^n$.

In particular, this hypothesis holds for the cohomology ring $H^*(G)$.

**Definition 6.2.** Let $A$ be a graded commutative $k$-algebra as in Hypothesis 6.1.
1. The Koszul complex $K(\zeta; A)$ of an element $\zeta \in A^n$ with $n > 0$ is by definition the following complex $(C^*, \delta)$:
   a) $C^0$ is the free $A$-module on one generator $u_\zeta$.
   b) $C^1$ is the free $A$-module on one generator $v_\zeta$.
   c) $C^i$ is zero otherwise.
   d) $\delta: C^i \to C^{i+1}$ sends $u_\zeta$ to $\zeta v_\zeta$ (for $i = 0$).

   Since $A$ is itself graded the Koszul complex is bigraded: that is, we may write $C^i = \bigoplus_{j \in \mathbb{Z}} C^{i,j}$ where for $x \in A^m$ we have $xu_\zeta \in C^{0,m}$ and $xv_\zeta \in C^{1,m-n}$. Then $\delta$ has bidegree $(1,0)$.

2. Let $\zeta_1, \ldots, \zeta_r$ be a sequence of homogeneous elements of $A^\ast$. If the prime $p$ is odd we shall suppose further\footnote{In order to avoid having to pay attention to signs.} that each $\zeta_i$ is an element of $A^{2\ast}$. The Koszul complex of this sequence is then defined by

$$\mathcal{K}(\zeta_1, \ldots, \zeta_r; A) := \mathcal{K}(\zeta_1; A) \otimes_A \cdots \otimes_A \mathcal{K}(\zeta_r; A).$$

This complex inherits a bigraded and the coboundary $\delta$ has bidegree $(1,0)$. Hence there are homology groups $H^{*,j}(\mathcal{K}(\zeta_1, \ldots, \zeta_r; A))$ for each $j \in \mathbb{Z}$.

The version of Carlson’s criterion in the following section differs very slightly from the original: we shall not require the parameters in Condition 6.6 to be even-dimensional if $p = 2$. The following lemma ensures that both versions of the criterion are equivalent.

**Lemma 6.3.** Let $A$ be a graded commutative $k$-algebra, as in Hypothesis 6.1. Let $\zeta_1, \zeta_2, \ldots, \zeta_r$ be a sequence of homogeneous elements of $A^\ast$ satisfying

$$H^{*,j}(\mathcal{K}(\zeta_1, \ldots, \zeta_r; A)) = 0 \text{ for every } j \geq 0. \quad (6.1)$$

Replacing $\zeta_1$ by $\zeta_1^2$ preserves this property $(6.1)$.

**Proof.** Set $\zeta := \zeta_1$ and write $(C^{*,*}, \delta)$ for the complex $\mathcal{K}(\zeta_2, \ldots, \zeta_r; A)$. Denote by $(K^{*,*}, \delta)$ the complex $\mathcal{K}(\zeta; A)$, and write $(L^{*,*}, \delta)$ for the complex $\mathcal{K}(\zeta^2; A)$.

We know that $H^{*,j}(K \otimes_A C) = 0$ for every $j \geq 0$ and have to prove the same statement for $H^{*,j}(L \otimes_A C)$. To simplify notation we write $u := u_\zeta$, $v := v_\zeta$, $u' := u_{\zeta^2}$, $v' := v_{\zeta^2}$ and $n := |\zeta|.$

Let $u' \otimes f + v' \otimes g$ be a cocycle in $Z^{ij}(L \otimes_A C)$ with $j \geq 0$. Hence $f$ lies in $C^{ij}$, $g$ lies in $C^{i-1,j+2n}$, and $\delta(u' \otimes f + v' \otimes g) = 0$. Consequently

$$\delta f = 0 \quad \text{and} \quad \zeta^2 f - \delta g = 0.$$

So $u \otimes \zeta f + v \otimes g \in (K \otimes_A C)^{i,j+n}$ is a cocycle and by assumption there are $a \in C^{i-1,j+n}$ and $b \in C^{i-2,j+2n}$ with $\delta(u \otimes a + v \otimes b) = u \otimes \zeta f + v \otimes g$. That is,

$$\delta a = \zeta f \quad \text{and} \quad \zeta a - \delta b = g.$$

But then $u \otimes f + v \otimes a \in (K \otimes_A C)^{ij}$ is another cocycle and bounds by assumption. So there are $c \in C^{i-1,j}$ and $e \in C^{i-2,j+m}$ satisfying $\delta(u \otimes c + v \otimes e) = u \otimes f + v \otimes a$. But then $u' \otimes c + v' \otimes (\zeta e + b)$ lies in $(L \otimes_A C)^{i-1,j}$ and satisfies $\delta(u' \otimes c + v' \otimes (\zeta e + b)) = u' \otimes f + v' \otimes g.$ \qed