Techniques for Nonsmooth Analysis on Smooth Manifolds I: Local Problems

Yu. S. Ledyaev[1] and Qiji J. Zhu[2]

[1] Department of Mathematics, Western Michigan University, Kalamazoo, MI 49008 and Steklov Institute of Mathematics, Moscow 117966, Russia. ledyaev@wmich.edu

[2] Department of Mathematics, Western Michigan University, Kalamazoo, MI 49008. zhu@wmich.edu

1 Introduction

Nonsmooth analysis, differential analysis for functions without differentiability, has witnessed a rapid growth in the past several decades stimulated by intrinsic nonsmooth phenomena in control theory, optimization, mathematical economics and many other fields. In the past several years many problems in control theory, matrix analysis and geometry naturally led to an increasing interest in nondifferentiable functions on smooth manifolds. Since a smooth manifold only locally resembles a Euclidean space and, in general, lacks of a linear structure, new techniques are needed to adequately address these problems. A number of results and techniques for dealing with such problems have emerged recently [8, 16, 18, 32]. The purpose of this paper is to report some useful techniques that we developed in the past several years for studying nonsmooth functions on smooth manifolds.

Many competing definitions of generalized derivatives have been developed for nonsmooth functions on Banach spaces [2, 9, 10, 21, 22, 24, 25, 29, 30, 31, 33, 34]. We choose to focus on a generalization of the Fréchet subdifferential and its related limiting subdifferentials. These objects naturally fit the variational technique that we use extensively (see [27] for a comprehensive introduction to the variational techniques in finite dimensional spaces). Using these generalized derivative concepts in dealing various problems involving nondifferentiable functions on smooth manifolds, we find three techniques particularly helpful. They are (i) using a chain rule to handle local problems, (ii) finding forms of essential results in nonsmooth analysis on Banach spaces that do not rely on the linear structure of such spaces so that they can be developed on a smooth manifold and (iii) using flows. We divide our paper in two parts. In this first part, we discuss local problems such as deriving necessary optimality conditions for local minimization problems and calculus rules for subdifferentials. For these problems, it is often efficient to use the

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following scheme. First use a local coordinate system of the smooth manifold to project the problem into a Euclidean space. Then use results in nonsmooth analysis that have been established in such a Euclidean space to derive results for this corresponding problem in the Euclidean space. Finally, use the local coordinate mapping and its induced mapping between the tangent or cotangent spaces to lift the results back to the original problem on the manifold. Results related to methods (ii) and (iii) will be discussed in the second part of the paper [17].

We will introduce notation and necessary preliminaries about smooth manifolds in the next section. In Section 3 we introduce subdifferentials for lower semicontinuous functions on smooth manifolds and the related concepts of normal cones to closed sets. In Sections 4 we discuss how to use a chain rule to handle nonsmooth problems on smooth manifolds that are local in nature by a chain rule.

2 Preliminaries and Notation

In this section we recall some pertinent concepts and results related to a smooth manifold. Our main references are [5, 20, 28].

Let M be an N-dimensional C^∞ complex manifold (paracompact Hausdorff space) with a C^∞ atlas $\{(U_a, \psi_a)\}_{a\in A}$. For each a, the set of N components $(x_a^1, ..., x_a^N)$ of ψ_a is called a local coordinate system on (U_a, ψ_a). A function $g : M \to \mathbb{R}$ is C^r at $m \in M$ if $m \in U_a$ and $g \circ \psi_a^{-1}$ is a C^r function in a neighborhood of $\psi_a(m)$. Here r is a nonnegative integer or ∞. As usual C^0 represents the collection of continuous functions. It is well known that this definition is independent on the coordinate systems. If g is C^∞ at all $m \in M$, then we say g is C^∞ on M. The collection of all C^∞ (resp., C^r) functions on M is denoted by $C^\infty(M)$ (resp., $C^r(M)$). A map $v : C^\infty(M) \to \mathbb{R}$ is called a *tangent vector* of M at m provided that, for any $f, g \in C^\infty(M)$, (1) $v(\lambda f + \mu g) = \lambda v(f) + \mu v(g)$ for all $\lambda, \mu \in \mathbb{R}$ and (2) $v(f \cdot g) = v(f)g(m) + f(m)v(g)$. The collection of all the tangent vectors of M at m form an (N-dimensional) vector space and is denoted by $T_m(M)$. The union $\bigcup_{m\in M}(m, T_m(M))$ forms a new space called the *tangent bundle* to M, and denoted by $T(M)$. The dual space of $T_m(M)$ is called the *cotangent space* of M at m, denoted by $T_m^*(M)$. The *cotangent bundle* to M then is $T^*(M) := \bigcup_{m\in M}(m, T_m^*(M))$. We will use π (resp., π^*) to denote the canonical projection on $T(M)$ (resp., $T^*(M)$) defined by $\pi(m, T_m(M)) = m$ (resp., $\pi^*(m, T_m^*(M)) = m$). A mapping $X : M \to T(M)$ is called a *vector field* provided that $\pi(X(m)) = m$. A vector field X is C^∞ (resp., continuous) at $m \in M$ provided so is $X(g)$ for any $g \in C^\infty$. If a vector field X is C^∞ (resp., continuous) for all $m \in M$ we say it is C^∞ (resp., continuous) on M. The collection of all C^∞ vector fields on M is denoted by $V^\infty(M)$.

In particular, if (U, ψ) is a local coordinate neighborhood with $m \in U$ and $(x^1, ..., x^N)$ is the corresponding local coordinate system on (U, ψ) then