Techniques for Nonsmooth Analysis on Smooth Manifolds II: Deformations and Flows

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1 Introduction

Nonsmooth analysis, differential analysis for functions without differentiability, has witnessed a rapid growth in the past several decades stimulated by intrinsic nonsmooth phenomena in control theory, optimization, mathematical economics and many other fields. In the past several years many problems in control theory, matrix analysis and geometry naturally led to an increasing interest in nondifferentiable functions on smooth manifolds. Since a smooth manifold is only locally resembles a Euclidean space and, in general, lacks of a linear structure, new techniques are needed for adequately address these problems. A number of results and techniques for dealing with such problems have emerged recently [6, 13, 15, 25].

This is the continuation of our paper [14] in this collection reporting some useful techniques that we developed in the past several years for studying nonsmooth functions on smooth manifolds. In this second part of the paper we discuss the following two issues. (a) Unlike a Banach space, a smooth manifold in general does not have a linear structure. Thus, many known results in Banach spaces that depend on the linear structure of such spaces cannot be directly extended to smooth manifolds. The extremal principles developed in [12, 17, 19], the nonconvex separation theorems discussed in [2, 27] and the multidirectional mean value theorems derived in [8, 9] are typical examples. For this kind of problem, we often need to look at ways to reformulate the original results in a form that does not depend on the linear structure. This reformulation then can be generalized to the setting of a smooth manifold. (b) Using flows determined by vector fields or multi-vector fields in the form of a differential inclusion in the tangent bundle of a manifold is often valuable in dealing with problems on smooth manifolds (both smooth and nonsmooth). For example, a geodesic in a Riemann manifold and its corresponding vector field is a natural replacement of a line segment and its generating vector in a

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Euclidean space. Thus, it is not surprising that flows, vector fields and multi-vector fields become the language for describing the analogue of a line or a convex set on a smooth manifold.

We will continue to use the notation and preliminaries from Part I of this paper [14] along with some additional preliminaries in the next section. In the rest of the paper we discuss the two techniques alluded to above and illustrate how to use them by sample results.

2 Preliminaries

We need some additional notations and preliminaries on Riemann manifolds. Again our main references are [5, 16, 22].

Recall that a mapping $g : T(M) \times T(M) \to \mathbb{R}$ is a C^∞ *Riemann metric* if

(1) for each m, $g_m(v, u)$ is a inner product on $T_m(M)$;
(2) if (U, ψ) is a local coordinate neighborhood around m with local coordinate system $(x^1, ..., x^N)$, then

$$g_{ij}(m) := g_m\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right) \in C^\infty(M).$$

One can check that (2) is independent of local coordinate systems. The manifold M together with the Riemann metric g is called a *Riemann manifold.* Since any paracompact manifold admits a positive-definite metric structure, in many cases we may assume that M is a Riemann manifold without significant loss of generality.

Let (M, g) be a Riemann manifold . For each $m \in M$, the Riemann metric induces an isomorphism between $T_m(M)$ and $T_m^*(M)$ by

$$v^* = g_m(v, \cdot) \quad (\langle v^*, u \rangle = g_m(v, u), \ \forall u \in T_m(M)).$$

Then we define norms on $T_m(M)$ and $T_m^*(M)$ by

$$\|v^*\|^2 = \|v\|^2 = g_m(v, v).$$

The following generalized Cauchy inequality is crucial: for any $v^* \in T_m^*(M)$ and $u \in T_m(M)$,

$$\langle v^*, u \rangle \le \|v^*\| \|u\|.$$

Let $r : [0, 1] \to M$ be a C^1 curve. The length of r is

$$l(r) = \int_0^1 \|r'(s)\| ds.$$

Let $m_1, m_2 \in M$. Denote the collection of all C^1 curves joining m_1 and m_2 by $C(m_1, m_2)$. Then the distance between m_1 and m_2 is defined by