Chapter 2

The Category of Graded Modules

2.1 Graded Modules

Throughout this section $R = \bigoplus_{\sigma \in G} R_\sigma$ is a graded ring of type $G$ for some fixed group $G$. A (left) $G$-graded $R$-module (or simple graded module) is a left $R$-module $M$ such that $M = \bigoplus_{x \in G} M_x$ where every $M_x$ is an additive subgroup of $M$, and for every $\sigma \in G$ and $x \in G$ we have $R_\sigma M_x \subseteq M_{\sigma x}$. Since $R_\sigma M_x \subseteq M_x$ we see that every $M_x$ is an $R_\sigma$-submodule of $M$. The elements of $\bigcup_{x \in G} M_x$ are called the homogeneous elements of $M$. A nonzero element $m \in M_x$ is said to be homogeneous of degree $x$, and we write $\deg(m) = x$. Every $m \in M$ can be uniquely represented as a sum $m = \sum_{x \in G} m_x$, with $m_x \in M_x$ and finitely many nonzero $m_x$. The nonzero elements $m_x$ in this sum are called the homogenous components of $m$. The set $\text{sup}(m) = \{ x \in G | m_x \neq 0 \}$ is called the support of $m$. We also denote by $\text{sup}(M) = \{ x \in G | M_x \neq 0 \}$ the support of the graded module $M$. If $\text{sup}(M)$ is a finite set (we denote by $\text{sup}(M) < \infty$) we say that $M$ is a graded module of finite support.

An $R$-submodule $N$ of $M$ is said to be a graded submodule if for every $n \in N$ all its homogenous components are also in $N$, i.e. : $N = \bigoplus_{\sigma \in G} (N \cap M_\sigma)$. For a graded submodule $N$ of $M$ we may define a quotient- (or factor-) structure on $M/N$ by defining a gradation as follows : $(M/N)_\sigma = M_\sigma + N/N$, for $\sigma \in G$. For an arbitrary submodule $N$ of a graded module $M$ we let $(N)_g$, resp. $(N)_g^9$, be the largest, resp. smallest, graded submodule of $M$ contained in $N$, resp. containing $N$. It is clear that $(N)_g$ equals the sum of all graded submodules of $M$ contained in $N$, while $(N)_g^9$ is the intersection of all graded submodules of $M$ containing $N$. We have : $(N)_g \subseteq N \subseteq (N)_g^9$. Of course, when $N$ itself is a graded submodule of $M$ then $(N)_g = N = (N)_g^9$. The set of $R$-submodules of a given module $M$ is usually denoted by $\mathcal{L}_R(M)$; in case $M$
is a graded $R$-module we look at the set $\mathcal{L}_R^g(M)$ of all graded $R$-submodules of $M$. It is easily verified that $\mathcal{L}_R(M)$ is a lattice with respect to the partial ordering given by inclusion and the operations $\cap$ and $\cup$; moreover $\mathcal{L}_R^g(M)$ is a sublattice of $\mathcal{L}_R(M)$.

Note that $(N)_g = \bigoplus_{\sigma \in G}(N \cap M_\sigma)$ is the submodule of $M$ generated by $N \cap h(M)$; on the other hand $(N)^g$ is the submodule of $M$ generated by the set $\bigcup_{n \in N}\{n_{\sigma}, \sigma \in G\}$, where $\{n_{\sigma}, \sigma \in G\}$ is the set of homogeneous components of $n \in N$.

From these observations it is also clear that an $R$-submodule $N$ of $M$ is a graded $R$-submodule if and only if $N$ has a set of generators consisting of homogeneous elements in $M$. All of the foregoing may be applied to left ideals $L$ of $R$ and two-sided ideals $I$ of $R$, in particular $(I)_g$ and $(I)^g$ are two-sided when $I$ is.

### 2.2 The category of Graded Modules

When the ring $R$ is graded by the group $G$ we consider the category $G - R$-gr, simply written $R$-gr if no ambiguity can arise, defined as follows. For the objects of $R$-gr we take the graded (left) $R$-modules and for graded $R$-modules $M$ and $N$ we define the morphisms in the graded category as:

$$\text{Hom}_{R-\text{gr}}(M, N) = \{f \in \text{Hom}_R(M, N), f(M_\sigma) \subset N_\sigma, \text{ for all } \sigma \in G\}$$

From the definition it is clear that $\text{Hom}_{R-\text{gr}}(M, N)$ is an additive subgroup of $\text{Hom}_R(M, N)$.

At this point it is useful though not really essential to have knowledge of a few basic facts in Category Theory; we include a short introduction in Appendix A.

The category $R$-gr has coproducts and products. Indeed, for a family of graded modules $\{M_i, i \in J\}$ a coproduct $S_J = \bigoplus_{\sigma \in G}S_\sigma$ may be given by taking $S_\sigma = \bigoplus_{i \in J}(M_i)_\sigma$ and a product $P_J$ may be obtained by taking $P_\sigma = \prod_{i \in J}(M_i)_\sigma$, so $P_J = \bigoplus_{\sigma \in G}\prod_{i \in J}(M_i)_\sigma$.

Since for any $f \in \text{Hom}_{R-\text{gr}}(M, N)$ we have a kernel, $\text{Ker}f$, and an image object, $\text{Im}f$, which are in $R$-gr and such that $M/\text{Ker}f \cong \text{Im}f$ are naturally isomorphic in this category, the category $R$-gr is an abelian category. It also follows that a graded morphism $f$ is a monomorphism, resp. epimorphism, in this category if and only if $f$ is injective, resp. surjective in the set theoretic sense.

In a straightforward way one may verify that $R$-gr satisfies Grothendieck’s axioms: $Ab3$, $Ab4$, $Ab3^*$, $Ab^*$ and also $Ab5$. 