Chapter 9

Application to Gradability

9.1 General Descent Theory

Let $\mathcal{A}$ and $\mathcal{B}$ be two categories and let $F : \mathcal{A} \to \mathcal{B}$ be a functor. In the case where $\mathcal{A}$ and $\mathcal{B}$ are additive categories, we assume that $F$ is an additive functor. Following the classical descent theory (see [115]) we can introduce a descent theory relative to the functor $F$. Consider an object $N \in \mathcal{B}$. We have the following problems.

i) \textit{Existence of $F$-descent objects} : does an object $M \in \mathcal{A}$ exist such that $N \cong F(M)$?

ii) \textit{Classification} : if such an $F$-descent object exists, classify (up to isomorphism) all objects $M$ for which $N \cong F(M)$.

9.1.1 Remarks

i) If the functor $F$ is an equivalence of categories, then for any $N \in \mathcal{B}$ there exists an unique (up to isomorphism) $F$-descent object.

ii) Assume that we have two functors $F : \mathcal{A} \to \mathcal{B}$ and $G : \mathcal{B} \to \mathcal{C}$. If for $Z \in \mathcal{C}$ there exists a $G \circ F$-descent object $X \in \mathcal{A}$, then $F(X)$ is a $G$-descent object for $Z$.

iii) If $F$ commutes with finite (arbitrary) coproducts or products, then any finite (arbitrary) coproduct or product of $F$-descent objects is also an $F$-descent object.

iv) Assume that $\mathcal{A}$ and $\mathcal{B}$ are abelian categories and $F$ is a faithful and exact functor which preserves isomorphisms. If $M \in \mathcal{B}$ is a simple object and $N \in \mathcal{A}$ is an $F$-descent object for $M$, then $N$ is simple in $\mathcal{A}$. Indeed, if $i : X \to N$ is a non-zero monomorphism in $\mathcal{A}$, then $F(i) : F(X) \to F(N) \cong M$ is a nonzero monomorphism,

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therefore $F(N) \simeq M$. Since $M$ is simple we see that $F(i)$ is an isomorphism, and then so is $i$. Thus $N$ is a simple object of $\mathcal{A}$.

### 9.1.2 Example

1. Let $R$ and $S$ be two rings and $\phi : R \to S$ be a ring morphism. Let $\mathcal{A} = R\text{-mod}$ and $\mathcal{B} = S\text{-mod}$ be the categories of modules. We have the following three natural functors:

- $S \otimes_R - : R\text{-mod} \to S\text{-mod}$ (the induced functor)
- $\text{Hom}_{R}(RS, -) : R\text{-mod} \to S\text{-mod}$ (the coinduced functor)
- $\phi_* : S\text{-mod} \to R\text{-mod}$ (the restriction of scalars)

When $S = l, R = k$ and $l$ is a commutative faithfully flat $k$-algebra, then the descent theory relative to the induced functor is exactly the classical descent theory.

2. Assume that $R \subseteq S$ is a ring inclusion. Then the descent theory relative to the functor $\phi_*$ (here $\phi : R \to S$ is the inclusion morphism) is exactly the problem of extending the module structure, i.e. of investigating whether for $M \in R\text{-mod}$ there exists a structure of an $S$-module on $M$ which by the restriction of scalars to $R$ gives exactly the initial $R$-module structure on $M$. In particular, if $S = \bigoplus_{\sigma \in G} S_{\sigma}$ is a $G$-strongly graded ring and $R = S_e$, we obtain the theory of extending modules given in Section 4.7.

3. Let $R = \bigoplus_{\sigma \in G} R_\sigma$ be a $G$-graded ring. We consider the forgetful functor $U : R\text{-gr} \to R\text{-mod}$. If $M \in R\text{-mod}$ and there exists an $U$-descent object $N \in R\text{-gr}$ for $M$, i.e. $U(N) \simeq M$, then $M$ is called a gradable module. If $G$ is a finite group, we can consider the smash product $\tilde{R} \# G$ and the natural morphism $\eta : R \to \tilde{R} \# G$ (see Chapter 7). By Proposition 7.3.10, $M \in R\text{-mod}$ is gradable if and only if $M$ has an extending relative to the morphism $\eta$.

Using the structure of gr-injective modules (Section 2.8) we have the following.

### 9.1.3 Proposition

Let $G$ be a finite group, $R$ a $G$-graded ring and $Q$ an injective $R$-module. The following assertions are equivalent.

i. $Q$ is gradable.

ii. There exists an injective $R_e$-module $N$ such that $Q \simeq \text{Coind}(N) = \text{Hom}_{R_e}(R, N)$ as $R$-modules.