Finiteness of Tate-Shafarevich groups

Let $E/F$ be an elliptic curve with split multiplicative reduction at $\infty$, with the notation of §2.1 of Chapter 2. In this chapter we extend the main result on the Tate conjecture for the elliptic surface corresponding to $E$ proved in [Br2] to general global fields of positive characteristic; we also eliminate some of the technical hypotheses contained therein.

The point is to prove the finiteness of the $l$-primary component of the Tate-Shafarevich group $\Sha(E/F)$ of the elliptic curve $E/F$ for at least one prime number $l$; this then implies both the finiteness of the group $\Sha(E/F)$ and the Tate conjecture for the corresponding elliptic surface by known results on étale cohomology (due to Artin, Tate, and Milne).

The contents of this chapter are the following. After some preliminaries in §7.1, Igusa’s determination of the galois action on torsion points of an elliptic curve over a global field of positive characteristic is presented in §§7.2-7.4, as well as some of its consequences. Further preliminaries are given in §§7.5-7.12 on the Tate conjecture.

The Heegner module $\mathcal{H}_{c,S}$ of the elliptic curve $E/F$ is introduced in §7.12; in §7.13 the determination of the galois invariants of the Heegner module (theorem 6.10.7) provides a homomorphism

$$\eta : \mathcal{H}_{c,S} \otimes S M(c) \rightarrow (\mathcal{H}_{c,S})^{G(c/0)}$$

(see proposition 7.13.4 for the notation) for the case where the primes components of $c$ are distinct, inert, and unramified in $K/F$ and where $S = \mathbb{Z}/l^n\mathbb{Z}$ for a suitable prime number $l$. The parametrisation of the elliptic curve $E$ by the Drinfeld modular curve $X_0^{\text{Drin}}(I)$, where $I$ is the conductor of $E$ without the component at $\infty$, provides the homomorphism

$$(\mathcal{H}_{c,S}^{(0)})^{G(c/0)} \rightarrow H^1(K, E)$$
whose domain is the galois invariant part of the Heegner module. If \( c \) is prime to the conductor \( I \), composing this map with \( \eta \) gives a homomorphism

\[
\mathcal{H}_{c,S} \otimes S M(c) \to H^1(K,E)
\]
whose image is a principal \( S \)-module generated by a cohomology class \( \delta(c) \) in \( H^1(K,E) \) (lemma 7.14.9, notation 7.14.10). It is these cohomology classes \( \delta(c) \), whose properties are considered in §7.14, that provide annihilators by duality (Tate-Poitou duality §§7.15-7.16 and Pontrjagin duality §7.17) of the Tate-Shafarevich group \( \prod(E/F) \) in §7.18; this suffices to prove the finiteness of the \( l \)-primary component of the Tate-Shafarevich groups for a set of prime numbers \( l \) of positive Dirichlet density.

### 7.1 Quasi-modules

In this section we introduce quasi-modules, trivial quasi-modules, and quasi-isomorphisms.

**Partially ordered sets and quasi-modules**

(7.1.1) Let \( A \) be a set equipped with a partial order written \( \succeq \), that is to say \( \succeq \) is reflexive, transitive, and verifies the condition: if \( x \succeq y \) and \( y \succeq x \) then \( x = y \).

Let \( A_{\mathrm{cat}} \) be the category associated to \( A \); that is, there is a bijection between \( A \) and the objects of \( A_{\mathrm{cat}} \) noted

\[
A \to \mathrm{Ob}(A_{\mathrm{cat}}), \quad x \to [x],
\]

and there is a unique arrow \([x] \to [y]\) between two objects \([x],[y]\) of \( A_{\mathrm{cat}} \) if and only if \( x \succeq y \).

#### 7.1.2. Examples.** The main examples of partially ordered sets \( A \) that we consider are these.

(i) Let \( R \) be a commutative ring. Let \( \{I_\lambda\}_{\lambda \in A} \) be a family of ideals of \( R \) partially ordered by inclusion; that is to say, for \( \lambda_1, \lambda_2 \in A \) we have \( \lambda_1 \succeq \lambda_0 \) if and only if \( I_{\lambda_1} \subseteq I_{\lambda_0} \).

(ii) Let \( A \) be the set

\[
\mathcal{P} = \{p^n | \text{ } p \text{ is a prime number and } n \in \mathbb{N}\}
\]

the set of all prime powers of \( \mathbb{N} \) where the partial order is given by divisibility: \( p^a \succeq q^b \) if and only if \( q^b | p^a \).