This chapter deals with more advanced aspects of almost commutative algebra: we begin with the definitions of Jacobson radical \( \text{rad}(A) \) of an almost algebra \( A \), of henselian pair and henselization of a pair \( (A, I) \), where \( I \subset A \) is an ideal contained in \( \text{rad}(A) \). These notions are especially well behaved when \( I \) is a right ideal (definition 5.1.5), in which case we can also prove a version of Nakayama’s lemma (lemma 5.1.7).

In section 5.3 we explain what is a linear topology on an \( A \)-module and an \( A \)-algebra; as usual, one is most interested in the case of \( I \)-adic topologies. In case \( A \) is \( I \)-adically complete and \( I \) is tight, we can show that the functor \( B \mapsto B/IB \) from almost finitely presented étale \( A \)-algebras to almost finitely presented étale \( A/I \)-algebras is an equivalence (theorem 5.3.27). For the proof we need some criteria to ensure that an \( A \)-algebra is unramified under various conditions: such results are collected in section 5.2, especially in theorem 5.2.12 and its corollary 5.2.15.

In section 5.5, theorem 5.3.27 is further generalized to the case where the pair \( (A, I) \) is tight henselian (see theorem 5.5.7, that also contains an analogous statement concerning almost finitely generated projective \( A \)-modules). The proof is a formal patching argument, which can be outlined as follows. First one reduces to the case where \( I \) is principal, say generated by \( f \), and since \( I \) is tight, one can assume that \( f \in m \); hence, we want to show that a given étale almost finitely presented \( A/fA \)-algebra \( B_0 \) lifts uniquely to an \( A \)-algebra \( B \) of the same type; in view of section 5.3 one can lift \( B_0 \) to an étale algebra \( B^\wedge \) over the \( f \)-adic completion \( A^\wedge \) of \( A \); on the other hand, the almost spectrum \( \text{Spec} A \) is a usual scheme away from the closed subscheme defined by \( I \), so we can use standard algebraic geometry to lift \( B^\wedge[f^{-1}] \) to an étale algebra \( B' \) over \( A[f^{-1}] \). Finally we need to show that \( B^\wedge \) and \( B' \) can be patched in a unique way over \( \text{Spec} A \); this amounts to showing that certain commutative diagrams of functors are 2-cartesian (proposition 5.5.6).

The techniques needed to construct \( B' \) are borrowed from Elkik’s paper [31]; for our purpose we need to extend and refine slightly Elkik’s results, to deal with non-noetherian rings. This material is presented in section 5.4; its usefulness transcends the modest applications to almost ring theory presented here.

The second main thread of the chapter is the study of the smooth locus of an almost scheme; first we consider the affine case: as usual, an affine scheme \( X \) over \( S := \text{Spec} A \) can be identified with the fpqc sheaf that it represents; then the smooth locus \( X_{\text{sm}} \) of \( X \) is a certain natural subsheaf, defined in terms of the cotangent complex \( L_{X/S} \). To proceed beyond simple generalities one needs to impose some finiteness conditions on \( X \), whence the definition of almost finitely presented scheme over \( S \). For such affine \( S \)-schemes we can show that the smooth locus enjoys a property which we could call almost formal smoothness. Namely, suppose that \( I \subset A \) is a tight ideal such that the pair \( (A, I) \) is henselian; suppose furthermore that \( \sigma_0 : S_0 := \text{Spec} A/I \rightarrow X \) is a section lying in the smooth locus of \( X \); in this situation it does not necessarily follow that \( \sigma_0 \) extends to a full section \( \sigma : S \rightarrow X \), however \( \sigma \) always exists if \( \sigma_0 \) extends to a section over some closed subscheme of the form \( \text{Spec} A/m_0 I \) (for a finitely generated subideal \( m_0 \subset m \)).
Next we consider quasi-projective almost schemes; if \( X \) is such a scheme, the invertible sheaf \( \mathcal{O}_X(1) \) defines a quasi-affine \( \mathbb{G}_m \)-torsor \( Y \to X \), and we define the smooth locus \( X_{\text{sm}} \) just as the projection of the smooth locus of \( Y \). This is presumably not the best way to define \( X_{\text{sm}} \), but anyway it suffices for the applications of section 5.8. In the latter we consider again a tight henselian pair \((A, I)\), and we study some deformation problems for \( G \)-torsors, where \( G \) is a closed subgroup scheme of \( \text{GL}_n \), defined over \( \text{Spec} \ A \) and fulfilling certain general assumptions (see (5.8.4)). For instance, theorem 5.8.21 says that every \( G \)-torsor over the closed subscheme \( \text{Spec} \ A/I \) extends to a \( G \)-torsor over the whole of \( \text{Spec} \ A \); the extension is however not unique, but any two such extensions are close in a precise sense (theorem 5.8.19): here the almost formal smoothness of the quasi-projective almost scheme \((\text{GL}_n/G)^a\) comes into play and accounts for the special quirks of the situation.

5.1. Henselian pairs.

**Definition 5.1.1.** Let \( A \) be a \( V^a \)-algebra. The Jacobson radical of \( A \) is the ideal \( \text{rad}(A) := \text{rad}(A^a) \) (where, for a ring \( R \), we have denoted by \( \text{rad}(R) \) the usual Jacobson ideal of \( R \)).

**Lemma 5.1.2.** Let \( R \) be a \( V \)-algebra, \( I \subset R \) an ideal. Then \( I^a \subset \text{rad}(R^a) \) if and only if \( mI \subset \text{rad}(R) \).

**Proof.** Let us remark the following:

Claim 5.1.3. If \( S \) is any ring, \( J \subset S \) an ideal, then \( J \subset \text{rad}(S) \) if and only if, for every \( x \in J \) there exists \( y \in J \) such that \((1 + x) \cdot (1 + y) = 1 \).

**Proof of the claim.** Suppose that \( J \subset \text{rad}(S) \) and let \( x \in J \); then \( 1 + x \) is not contained in any maximal ideal of \( S \), so it is invertible. Find some \( u \in S \) with \( u \cdot (1 + x) = 1 \); setting \( y := u - 1 \), we derive \( y = -x - xy \in J \). Conversely, suppose that the condition of the claim holds for all \( x \in J \). Let \( x \in J \); we have to show that \( x \in \text{rad}(S) \). If this were not the case, there would be a maximal ideal \( q \subset S \) such that \( x \notin q \); then we could find \( a \in S \) such that \( x \cdot a \equiv -1 \pmod{q} \), so \( 1 + x \cdot a \in q \), especially \( 1 + a \cdot x \) is not invertible, which contradicts the assumption. \( \diamond \)

Let \( \phi : mI \to R \to R^a \) be the natural composed map.

Claim 5.1.4. \( \text{Im} \phi \) is an ideal of \( R^a \) and \( mI \subset \text{rad}(R) \) if and only if \( \text{Im} \phi \subset \text{rad}(R^a) \).

**Proof of the claim.** The first assertion is easy to check, and clearly we only have to verify the ”if” direction of the second assertion, so suppose that \( \text{Im} \phi \subset \text{rad}(R^a) \). Notice that \( \text{Ker} \phi \) is a square-zero ideal of \( R \). Then, using claim 5.1.3, we deduce that for every \( x \in m \cdot I \) there exists \( z \in m \cdot I \) such that \((1 + x) \cdot (1 + z) = 1 + a \), where \( a \in \text{Ker} \phi \), so \( a^2 = 0 \). Consequently \((1 + x) \cdot (1 + z) \cdot (1 - a) = 1 \), so the element \( y := z - a - z \cdot a \) fulfills the condition of claim 5.1.3. \( \diamond \)

It is clear that \( I^a \subset \text{rad}(R^a) := \text{rad}(R^a)^a \) if and only if \( \text{Im} \phi \subset \text{rad}(R^a) \), so the lemma follows from claim 5.1.4. \( \square \)