Certain alternating sums of operators

In this chapter we will deal with alternating sums

\[
\begin{align*}
H^\frac{1}{2} XK^\frac{1}{2} \\
H^\frac{1}{2} XK^\frac{3}{2} - H^\frac{3}{2} XK^\frac{1}{2} \\
H^\frac{1}{2} XK^\frac{1}{2} - H^\frac{1}{2} XK^\frac{3}{2} + H^\frac{3}{2} XK^\frac{1}{2} \\
H^\frac{1}{2} XK^\frac{1}{2} - H^\frac{1}{2} XK^\frac{3}{2} + H^\frac{3}{2} XK^\frac{3}{2} - H^\frac{1}{2} XK^\frac{1}{2} \\
\cdots
\end{align*}
\]

\[
\begin{align*}
XX - HX \\
XX - H^\frac{1}{2} XK^\frac{1}{2} + HX \\
XX - H^\frac{1}{2} XK^\frac{3}{2} + H^\frac{3}{2} XK^\frac{1}{2} - HX \\
XX - H^\frac{1}{2} XK^\frac{1}{2} + H^\frac{3}{2} XK^\frac{3}{2} - H^\frac{1}{2} XK^\frac{1}{2} + HX \\
\cdots
\end{align*}
\]

and investigate behavior of unitarily invariant norms of these operators such as mutual comparison, uniform bounds (independent of \( n, m \)), monotonicity and so on (in §8.2 and §8.3). For convenience we set

\[
\begin{align*}
A(n) &= \sum_{k=1}^{n} (-1)^{k-1} H_{\frac{k}{n+1}} XK_{\frac{n+1-k}{n+1}} & (n = 1, 2, 3, \cdots), \\
B(m) &= \sum_{k=0}^{m-1} (-1)^{k} H_{\frac{k}{m-1}} XK_{\frac{m-1-k}{m-1}} & (m = 2, 3, 4, \cdots),
\end{align*}
\]

and these notations will be kept throughout. We note

\[
B(m) = \begin{cases} 
HX + XK - A(m-2) & \text{for } m = 3, 5, 7, \cdots, \\
-HX + XK - A(m-2) & \text{for } m = 4, 6, 8, \cdots.
\end{cases} \quad (8.1)
\]

The nature of the above two series of operators depends strongly on parities of \( n \) and \( m \), and it is quite obvious that we will have to treat odd and even cases separately.
8.1 Preliminaries

For $n = 1, 2, \cdots$ and $m = 2, 3, \cdots$ we set

$$a_n(s, t) = \sum_{k=1}^{n} (-1)^{k-1} s^{k-1} t^{n-k}$$
and

$$b_m(s, t) = \sum_{k=0}^{m-1} (-1)^{k} s^{m-k} t^{k}$$

$(s, t \geq 0)$ as scalar “means” corresponding to $A(n)$ and $B(m)$. For $s, t > 0$ we compute

$$a_n(s, t) = \frac{s^{n+1}}{1 + \left(\frac{s}{t}\right)^{\frac{1}{n+1}}} \left(1 - (-1)^n \left(\frac{s}{t}\right)^{\frac{n}{n+1}}\right) = \frac{t^{1/n+1}}{1 + \left(\frac{s}{t}\right)^{\frac{1}{n+1}}} \left(1 - (-1)^n \left(\frac{s}{t}\right)^{\frac{n}{n+1}}\right)$$

$$= \left(\frac{s}{t}\right)^{\frac{1}{n+1}} \times \frac{\left(\frac{s}{t}\right)^{\frac{-1}{n+1}} - (-1)^n \left(\frac{s}{t}\right)^{\frac{1}{n+1}}}{\left(\frac{s}{t}\right)^{\frac{-1}{n+1}} + \left(\frac{s}{t}\right)^{\frac{1}{n+1}}}$$

$$= \left(st\right)^{\frac{1}{n+1}} \times \frac{\left(\frac{s}{t}\right)^{\frac{-1}{n+1}} - (-1)^n \left(\frac{s}{t}\right)^{\frac{1}{n+1}}}{\left(\frac{s}{t}\right)^{\frac{-1}{n+1}} + \left(\frac{s}{t}\right)^{\frac{1}{n+1}}}$$

Note that the denominator can be always expressed in terms of the hyperbolic cosine function while for the numerator the hyperbolic sine function is also needed for $n$ even. Exactly the same computations yield

$$b_m(s, t) = (st)^{\frac{1}{m+1}} \times \frac{\left(\frac{s}{t}\right)^{\frac{-1}{m+1}} - (-1)^m \left(\frac{s}{t}\right)^{\frac{1}{m+1}}}{\left(\frac{s}{t}\right)^{\frac{-1}{m+1}} + \left(\frac{s}{t}\right)^{\frac{1}{m+1}}}$$

These formulas will be freely and repeatedly used. We note the homogeneity

$$a_n(rs, rt) = ra_n(s, t), \quad b_m(rs, rt) = rb_m(s, t)$$

(with $r \geq 0$) and

$$a_n(t, s) = (-1)^{n+1} a_n(s, t), \quad b_m(t, s) = (-1)^{m+1} b_m(s, t)$$

(see Proposition 8.2, (iii)).

We will repeatedly make use of the positive definiteness of the following functions (see §6.3, 1):

$$\frac{1}{\cosh(\alpha x)}, \quad \frac{\cosh(\beta x)}{\cosh(\alpha x)}, \quad \frac{\sinh(\beta x)}{\sinh(\alpha x)}$$

with $0 < \beta < \alpha$ (as was done in preceding chapters). The next observation is also useful.