A New Class of Binary CSPs for which Arc-Consistency Is a Decision Procedure

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Abstract. In this report we introduce a new hybrid class for which arc-consistency is a decision procedure. This new hybrid class includes infinitely many instances whose tractability is not assured by any tractable language or structural restriction, and strongly motivates the search for a unifying principle for the tractable constraint classes decided by arc-consistency.

1 Introduction

The class of Constraint Satisfaction Problems (CSPs) is NP-hard. However, there are certain restrictions that make it tractable. These tractable classes have a polynomial decision procedure.

In this report we concentrate on those classes of constraint satisfaction problem instances for which arc-consistency ($k = 2$) is a decision procedure. We will define a new hybrid class of binary constraint problem instances with a non-Boolean domain for which arc-consistency is a decision procedure.

A constraint satisfaction problem instance (CSP) consists of a set of variables which have to be assigned values from some domain. The set of allowed values is restricted for certain subsets of the variables.

Definition 1. A CSP, is a triple $\langle V, D, C \rangle$, where:

- $V$ is a finite set of variables;
- $D$ is a finite set called the domain of $P$;
- $C$ is a set of constraints. Each constraint $c \in C$ is a pair $c = \langle \sigma, \rho \rangle$ where $\sigma = \langle v_1, \ldots, v_k \rangle$ is a list of variables from $V$, called the constraint scope, and $\rho$ is a subset of $D^k$ called the constraint relation.

A solution to $P = \langle V, D, C \rangle$ is an assignment $s$ of a value in $D$ to each variable $v$ such that, for every constraint $\langle \sigma, \rho \rangle$, the projection of $s$ onto $\sigma$ is contained in $\rho$. The set of solutions to $P$ is denoted $\text{Sol}(P)$.

The general CSP (decision) problem is NP-complete [6]. Naturally we want to identify subproblems for which polynomial algorithms exist.

Most tractability results rely on restricting either the structure, or the underlying language of the problem instances. We now define these concepts.
Definition 2. The structure of $P = \langle V, D, C \rangle$ is a hypergraph $\langle V, E \rangle$ whose vertexes $V$ are the set of variables of $P$, and whose hyperedges $E$ are the sets defined by the scopes of the constraints of $P$. That is:

$$E = \{\{x_1, \ldots, x_k\} \mid \exists \langle x_1, \ldots, x_k, \rho \rangle \in C, \}$$

A constraint language is a set of relations. The language of $P = \langle V, D, C \rangle$ is the set, $\Gamma_P$, of constraint relations occurring in $P$. That is:

$$\Gamma_P = \{\rho \mid \exists \langle \sigma, \rho \rangle \in C\}$$

Lastly, we need to define what we mean when we say a CSP is arc-consistent.

Definition 3. Let $P = \langle V, D, C \rangle$ be a CSP instance.

For any subset $W \subseteq V$ the restriction of $P$ to $W$, denoted $P^*_W$ is the instance with variables $W$ and domain $D$, where the constraints are obtained from the constraints of $P$ by eliminating all the constraints with scope not contained in $W$. That is, $P^*_W = \langle W, D, C' \rangle$ where $\langle \sigma, \rho \rangle \in C'$ if and only if $\langle \sigma, \rho \rangle \in C, \sigma \subseteq W$.

We say that $P$ is $(j, k)$-consistent ($0 \leq j \leq k$) if, for any sets of variables $W, W'$ with $W \subseteq W' \subseteq V$, containing at most $j$ and $k$ variables respectively, any solution to $P^*_W$ can be extended to a solution to $P^*_W'$.

A problem is arc-consistent if it is $(1, 2)$-consistent.

For every CSP, $P$, there is a unique CSP $A(P)$ which is arc-consistent and has the same set of solutions as $P$. The problem of determining $A(P)$ for any $P$ is polynomial.

2 When Arc-Consistency Is Enough

Sometimes establishing arc-consistency results in domain wipe-out. By this we mean that $A(P)$ has some variable with an empty unary constraint. In this case it is clear that $P$ cannot be solved.

We say that arc-consistency is a decision procedure for a class $C$ of CSPs if every CSP in $C$ either has a solution or arc-consistency results in domain wipe-out. Such a class is clearly tractable.

We have a characterisation of those structures for which arc-consistency is a decision procedure. We also have a characterisation of those languages for which there is some $k > 1$ for which $(1, k)$-consistency is a decision procedure.

Theorem 1. Let $\mathcal{H}$ be a class of hypergraphs. The class of CSPs whose structure is in $\mathcal{H}$ has arc-consistency as a decision procedure exactly when all the dual graphs are trees.

The result about $(1, k)$-consistency requires a definition.

\[1\] We assume an explicit representation of constraints as a set of allowed labellings. For the uniqueness of $A(P)$ we assume that we merge unary constraints.