Let $\mathfrak{A}$ be a Banach algebra, and let $\mathcal{E}$ be a class of Banach $\mathfrak{A}$-bimodules. If $\mathcal{E}$ is the class of all dual Banach $\mathfrak{A}$-bimodules, then $\mathfrak{A}$ is amenable if $H^1(\mathfrak{A}, E) = \{0\}$ for all $E \in \mathcal{E}$.

What happens, if we choose another class $\mathcal{E}$ of modules? What if $\mathcal{E}$ is the class of all Banach $\mathfrak{A}$-bimodules? What if $\mathcal{E}$ consists of all finite-dimensional Banach $\mathfrak{A}$-bimodules, or of all finitely generated Banach $\mathfrak{A}$-bimodules, or just of $\mathfrak{A}$ or $\mathfrak{A}^*$ alone? For each such class $\mathcal{E}$, we can define a corresponding notion of amenability. The question, is, of course, whether the class of Banach algebras we single out through such a definition is a good one: Do we get both strong theorems and a sufficient number of interesting examples?

We discuss two such notions of amenability in this chapter: super-amenability ($\mathcal{E}$ is the class of all Banach $\mathfrak{A}$-bimodules) and weak amenability ($\mathcal{E} = \{\mathfrak{A}^*\}$).

Two more classes of Banach algebras we deal with in this chapter are the biprojective and the biflat Banach algebras. Although these Banach algebras can also be characterized through an appropriate class $\mathcal{E}$ of bimodules, we define them via the existence of certain module homomorphisms.

Finally, we study Connes-amenability — a notion of amenability that only makes sense for Banach algebras which are dual spaces. As we shall see in Chapter 6, Connes-amenability is the right notion of amenability for von Neumann algebras.

This chapter is less self-contained than the previous ones: We need some basic facts from the theory of von Neumann algebras, all of which can be found in [K–R] and [Sak], as well as Haagerup’s non-commutative Grothendieck inequality ([Haa 3]); we also require some background on the representation theory of Banach algebras and on Banach algebras with non-zero socle, for which we refer to [B–D] and to [Pal].

### 4.1 Super-amenability

In Definition 2.1.9, we used the first Hochschild cohomology group of a Banach algebra with coefficients in a dual module to define amenable Banach algebras. Let’s consider the following variant:

**Definition 4.1.1** A Banach algebra $\mathfrak{A}$ is called super-amenable if $H^1(\mathfrak{A}, E) = \{0\}$ for every Banach $\mathfrak{A}$-bimodule $E$.

Certainly, every super-amenable Banach algebra is amenable. But what about the converse?
Many of the basic results for amenable Banach algebras have analogues for super-amenable Banach algebras. The proofs are often similar and, in fact, are even easier in the super-amenable situation. We thus leave them to the reader as a series of exercises.

**Exercise 4.1.1** Show that every super-amenable Banach algebra is unital.

**Exercise 4.1.2** Let $\mathfrak{A}$ be a super-amenable Banach algebra. Show that $\mathcal{H}^n(\mathfrak{A}, E) = \{0\}$ for all $n \in \mathbb{N}$.

Remember the definition of a diagonal? If not, go back to Exercise 2.2.1.

**Exercise 4.1.3** Show that a Banach algebra is super-amenable if and only if it has a diagonal. Conclude that, for $n_1, \ldots, n_k \in \mathbb{N}$, the algebra $M_{n_1} \oplus \cdots \oplus M_{n_k}$ is super-amenable.

**Exercise 4.1.4** Prove the following hereditary properties for super-amenability:

(i) If $\mathfrak{A}$ is a super-amenable Banach algebra, $\mathfrak{B}$ is a Banach algebra, and $\theta : \mathfrak{A} \to \mathfrak{B}$ is a homomorphism with dense range, then $\mathfrak{B}$ is super-amenable.

(ii) If $\mathfrak{A}$ is a super-amenable Banach algebra, and $I$ is a closed ideal of $\mathfrak{A}$, then $I$ is super-amenable if and only if it has an identity and if and only if it is complemented in $\mathfrak{A}$.

(iii) If $\mathfrak{A}$ is a Banach algebra and $I$ is a closed ideal of $\mathfrak{A}$ such that both $I$ and $\mathfrak{A}/I$ are super-amenable, then $\mathfrak{A}$ is super-amenable.

(iv) If $\mathfrak{A}$ and $\mathfrak{B}$ are super-amenable Banach algebras, then $\mathfrak{A} \hat{\otimes} \mathfrak{B}$ is super-amenable.

Is there an analogue of Proposition 2.3.17 for super-amenable Banach algebras?

**Exercise 4.1.5** Let $\mathfrak{A}$ be a super-amenable Banach algebra. Show that every admissible, short, exact sequence

$$
\{0\} \to F \to E \to E/F \to \{0\}
$$

of Banach $\mathfrak{A}$-modules (left-, right- or bi-) splits.

**Exercise 4.1.6** Let $\mathfrak{A}$ be a unital Banach algebra.

(i) Show that the short, exact sequence

$$
\{0\} \to \ker \Delta \to \mathfrak{A} \hat{\otimes} \mathfrak{A} \xrightarrow{\Delta} \mathfrak{A} \to \{0\} \quad (4.1)
$$

of Banach $\mathfrak{A}$-bimodules is admissible.

(ii) Show that the following are equivalent:

(a) $\mathfrak{A}$ is super-amenable.

(b) (4.1) splits.

Hence, super-amenable Banach algebras seem to have very nice properties — nicer than amenable Banach algebras. So, why don’t we study super-amenable Banach algebras in the first place? The answer to that question is that Definition 4.1.1 is so strong that, although it allows for nice theorems, all known examples are dull; if there are any non-dull examples, they have to be extremely pathological in terms of Banach space geometry.

We prepare the ground with the following proposition, for which we require some background on the representation theory of Banach algebras ([B–D, Chapter III]):