5. Semi-Riemannian Osserman Manifolds

5.1 Nonsymmetric Semi-Riemannian Osserman Manifolds of Arbitrary Signature

A positive solution to the Osserman problem in the Riemannian setting was provided under some hypothesis on the dimension of the manifold or on the existence of special additional structures on the manifold. In this subsection we will point out the nonexistence of analogous results in the general semi-Riemannian setting. First of all we know the existence of nonsymmetric semi-Riemannian Osserman manifolds with metric tensors of arbitrary signature \((\nu, \eta), \nu, \eta > 1\). For, let \((M_1, g_1)\) and \((M_2, g_2)\) be semi-Riemannian manifolds of dimensions \(n_1\) and \(n_2\), respectively. The product \(M = M_1 \times M_2\) furnished with the product metric tensor \(g = g_1 \oplus g_2\) is also a semi-Riemannian manifold. Moreover, for each vector field \(X = (X_1, X_2)\) tangent to \(M\), where \(X_1\) and \(X_2\) are vector fields on \(M_1\) and \(M_2\), respectively, the characteristic polynomial of the operator \(R_X = R(\cdot, X)X\) on \(TM\) satisfies

\[
p_{\lambda}(R_X) = \det(R_X - \lambda I_{n_1+n_2}) = \det(R_X^{(1)} - \lambda I_{n_1})\det(R_X^{(2)} - \lambda I_{n_2})
\]

where \(R^{(1)}\) and \(R^{(2)}\) are the curvature tensors of \((M_1, g_1)\) and \((M_2, g_2)\), respectively, and \(R^{(1)}_X = R^{(1)}(\cdot, X_1)X_1\) on \(TM_1\) and \(R^{(2)}_X = R^{(2)}(\cdot, X_2)X_2\) on \(TM_2\).

Now, it follows that the product manifold \((M, g)\) is Osserman if both factors \((M_1, g_1)\) and \((M_2, g_2)\) are Osserman with vanishing eigenvalues of their Jacobi operators. (Note that, in the Riemannian case, a pointwise Osserman manifold is flat if it is locally reducible \([9, \text{Lemma 2.2}]\).)

Therefore, if \((N, g(f_1, f_2))\) is one of the examples constructed in Section 4.1, then the product manifold \(\mathbb{R}^n \times N\) endowed with the product metric tensor \(g = g(\nu-2, \eta-2) \oplus g(f_1, f_2)\) is a semi-Riemannian manifold with metric tensor of signature \((\nu, \eta)\) that is Osserman but not locally symmetric, where \(g(\nu-2, \eta-2)\) denotes the semi-Euclidean metric tensor of signature \((\nu-2, \eta-2)\), \(\nu, \eta \geq 2\).

Remark 5.1.1. Note that, at each point of \((\mathbb{R}^n \times N, g)\) there exist unit vectors whose Jacobi operators have different minimal polynomials. Indeed, for \(x = (x_1, 0)\), the associated Jacobi operator vanishes identically and thus

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the minimal polynomial of $R_x$ becomes $m_\lambda(R_x) = \lambda$. However, since $R_x$ is nonzero for $x = (0, x_2)$, where $(N, g_{(f_1, f_2)})$ is chosen as in Remark 4.1.2, its minimal polynomial is $m_\lambda(R_x) = \lambda^s$, $s = 1, 2, 3$, depending on the point and the metric tensor $g_{(f_1, f_2)}$ defined on $N$. This shows that the Jacobi operators may be diagonalizable for some directions and may not be diagonalizable for other directions at a point.

**Remark 5.1.2.** For any locally symmetric semi-Riemannian Osserman manifold with $p_\lambda(R_x) = \lambda^4$ and $m_\lambda(R_x) = \lambda^2$ (cf. Theorem 4.2.7), the product manifolds constructed above are also locally symmetric. Note that, even in this case the minimal polynomial of the Jacobi operator has nonconstant roots, counting multiplicities, at each point. This shows the existence of symmetric Osserman manifolds that are not Jordan-Osserman.

### 5.1.1 Indefinite Kähler Osserman Manifolds

In Chapter 1, it is shown that the assumption of additional structures on a Riemannian Osserman manifold led to affirmative conclusions in the solution to the Osserman problem. Among such structures, the Kähler ones are well-understood (cf. Section 2.2.1.) At this point, it is important to recall the special role played by the holomorphic sectional curvature in Lemma 2.2.1. Hence one may expect to obtain some similar results in the semi-Riemannian case. For that, a closer examination of the holomorphic sectional curvature is needed.

The holomorphic sectional curvature is a real-valued function defined on the unit sphere bundle of an almost Hermitian manifold with Riemannian metric tensor and hence it is bounded at each point. However, for indefinite metric tensors, such a statement is no longer true, that is, the holomorphic sectional curvature of an indefinite almost Hermitian manifold is bounded (from above and below) at a point if and only if it is constant at that point [4], [27]. Therefore, one may expect to obtain affirmative conclusions in the study of the Osserman problem by imposing weak boundedness conditions on the holomorphic sectional curvature of an indefinite Kähler manifold. For example, one may restrict the condition of boundedness (from above and below) on curvature of holomorphic planes of signature $(++)$ or $(-)$. However this also yields constant holomorphic sectional curvature [27], [100]. Therefore a strictly weaker assumption on the holomorphic sectional curvature of a Kähler manifold is to assume that the curvature is bounded from below (or from above) on the holomorphic planes of signature $(+, +)$, and from above (or from below) on the holomorphic planes of signature $(-, -)$. This condition is equivalent to the vanishing Bochner tensor of the manifold [29], [100]. Now, since any Osserman manifold is Einstein, it immediately follows again that the indefinite Kähler Osserman manifolds whose curvature is bounded from below (or from above) on the holomorphic planes of signature $(+, +)$, and