6. Generalizations and Osserman-Related Conditions

In this chapter we briefly expose some concepts related to the Osserman problem. The exposition will not be so exhaustive as in the previous chapters since our purpose in this chapter is to point out a number of different problems related to the algebraic properties of the curvature tensors of semi-Riemannian manifolds.

As a generalization of the Jacobi operators, we first consider the generalized Jacobi operator associated to a $k$-plane at each point of a semi-Riemannian manifold. The investigation of the spectral properties of such operators is made in Section 6.1.

An attempt to generalize Osserman condition to affine differential geometry is subject to an additional difficulty that the unit sphere bundle cannot be defined. Therefore, any affine Osserman manifold necessarily has zero eigenvalues for the Jacobi operators. However, such manifolds are not necessarily flat and their properties are investigated in Section 6.2. As an application, other semi-Riemannian Osserman metric tensors are also constructed in the cotangent bundle of a torsion-free affine Osserman manifold (cf. §6.2.3.)

In Section 6.3 we discuss the conjecture of isoparametric geodesic spheres and its relation to the Osserman problem and Lichnerowicz conjecture on harmonic manifolds. A condition on the constancy of the eigenvalues of the Jacobi operators along geodesics is discussed in Section 6.4.

Finally, Section 6.5 is devoted to the so-called IP-spaces. They are Riemannian manifolds where the skew-symmetric curvature operator $R(x,y)$ has constant eigenvalues.

6.1 Semi-Riemannian Generalized Osserman Manifolds

The notion of generalized Osserman manifold is due to Stanilov and Videv [133], who originally investigated such a condition for the 4-dimensional case. Later on, their results were generalized by Gilkey [67] to arbitrary dimensions and extended to semi-Riemannian geometry in [76].

We start with the Riemannian case. Let $Gr_k(T_pM)$ be the Grassmannian of $k$-planes in $T_pM$ of a Riemannian manifold $(M,g)$. For each $E \in Gr_k(T_pM)$, let $J(E)$ denote the generalized Jacobi operator.
\[ J(E) = R(\cdot, x_1)x_1 + \cdots + R(\cdot, x_k)x_k, \]

where \( \{x_1, \ldots, x_k\} \) is an orthonormal basis for \( E \). (Note that \( J(E) \) is independent of the choice of orthonormal basis.)

**Definition 6.1.1.** A Riemannian manifold \((M, g)\) is called \( k \)-Osserman at \( p \in M \) if the characteristic polynomial of \( J(E) \) is independent of \( E \in \text{Gr}_k(T_pM) \), that is, the eigenvalues of \( J(E) \) counted with multiplicities are constant for every \( E \in \text{Gr}_k(T_pM) \). Also, \((M, g)\) is called globally \( k \)-Osserman if the characteristic polynomial of \( J(E) \) is independent of \( E \in \bigcup_{p \in M} \text{Gr}_k(T_pM) \).

Note here that any Riemannian Osserman manifold is 1-Osserman and moreover, real space forms are \( k \)-Osserman for all \( k \). However, there exist Riemannian Osserman manifolds which are not \( k \)-Osserman. On the other hand, there is a certain kind of duality between the concepts above [67], [133] as pointed out in (2) below.

**Theorem 6.1.1.** [67] Let \((M, g)\) be an \( n \)-dimensional Riemannian manifold.

1. If \((M, g)\) is \( k \)-Osserman at \( p \in M \) then \((M, g)\) is Einstein at \( p \in M \).
2. If \((M, g)\) is \( k \)-Osserman at \( p \in M \) then \((M, g)\) is \((n - k)\)-Osserman at \( p \in M \).
3. If \((M, g)\) is \( k \)-Osserman at \( p \in M \) and \( 2k \leq n \) then \((M, g)\) is 2-stein at \( p \in M \).

**Proof.** (1): Let \( \{x_1, x_2\} \) be orthonormal vectors in \( T_pM \) and let \( \{x_i, i = 1, \ldots, n\} \) be an extension of \( \{x_1, x_2\} \) to an orthonormal basis for \( T_pM \). Now let \( E \) and \( \tilde{E} \) be the \( k \)-planes defined by \( E = \text{span}\{x_1, x_3, \ldots, x_{k+1}\} \) and \( \tilde{E} = \text{span}\{x_2, x_3, \ldots, x_{k+1}\} \). Since \((M, g)\) is assumed to be \( k \)-Osserman at \( p \in M \), \( J(E) = J(\tilde{E}) \), and thus

\[
\text{trace}\{R_{x_1} + R_{x_3} + \cdots + R_{x_{k+1}}\} = \text{trace}\{R_{x_2} + R_{x_3} + \cdots + R_{x_{k+1}}\},
\]

and it follows that \( \text{Ric}(x_1, x_1) = \text{Ric}(x_2, x_2) \). Thus \((M, g)\) is Einstein at \( p \in M \).

(2): Let \( E \) be a \( k \)-plane in \( T_pM \) and \( E^\perp \) be the orthogonal \((n - k)\)-plane to \( E \) in \( T_pM \). Choose orthonormal bases \( \{x_i, i = 1, \ldots, k\} \) and \( \{x_i, i = k + 1, \ldots, n\} \) for \( E \) and \( E^\perp \), respectively. Then, since \((M, g)\) is Einstein by (1),

\[
J(E) + J(E^\perp) = \sum_{i=1}^{n} R(\cdot, x_i)x_i = \frac{S_c}{n} \text{Id},
\]

where \( S_c \) denotes the scalar curvature. Therefore, the eigenvalues of \( J(E) \) determine the eigenvalues of \( J(E^\perp) \) and hence the claim follows.

(3): To prove that \((M, g)\) is 2-stein at \( p \in M \), we need to show that

\[
\text{trace}R^{(2)}_{x_1} = \text{trace}R^{(2)}_{x_2}
\]

for any orthonormal vectors \( x_1, x_2 \in T_pM \). Let us