Order-$k$ Voronoi diagrams, $k$-sections, and $k$-sets

Dominique Schmitt and Jean-Claude Spehner

Laboratoire MAGE, Université de Haute-Alsace
4, rue des Frères Lumière, 68093 Mulhouse Cedex, France
{D.Schmitt, JC.Spehner}@univ-mulhouse.fr

Abstract. In this paper we characterize all-dimensional faces of order-$k$ Voronoi diagrams. First we introduce the notion of $k$-section to give a precise definition of these faces. Then, we characterize the unbounded faces by extending the classical notion of $k$-set. Finally, by studying some relations between $k$-sections, we give a new proof of the size of order-$k$ Voronoi diagrams in the plane.

1 Introduction

Let $S$ be a set of $n$ sites in the $d$-dimensional Euclidean space $E$ and let $P$ be a subset of $k$ sites of $S$ ($k \in \{1, \ldots, n-1\}$). The set of points of $E$ closer to each site of $P$ than to any other site of $S$ is either empty or a region of $E$. In this later case the region is called the order-$k$ Voronoi region of $P$. The set of order-$k$ Voronoi regions of all subsets of $k$ sites of $S$ is called the order-$k$ Voronoi diagram of $S$.

This diagram has first been treated by Miles [6] and has been introduced in computational geometry by Shamos and Hoey [10]. The construction of the order-$k$ Voronoi diagram leads to efficient algorithms in various applications such as $k$ nearest neighbor search, data clustering, ... In the numerous papers that have been published on the subject [2] [7], only order-$k$ Voronoi regions are studied. However, the boundaries of these regions intersect and, in order to study the partition of $E$ generated by the order-$k$ Voronoi diagram, it is necessary to consider independently all the $i$-dimensional faces of the diagram, $i \in \{0, \ldots, d\}$.

In this paper we show how the notion of $k$-section can be used to explicitly define the faces of the order-$k$ Voronoi diagram. The adjacency relations between these faces are also determined by a simple relation between their defining $k$-sections. Furthermore we show how the well known notion of $k$-set which characterizes unbounded order-$k$ Voronoi regions can be generalized to characterize all-dimensional unbounded order-$k$ Voronoi faces. We give all these results in the most general case where any number of sites may be coplanar or cospherical.

Lee [5] has shown that the size of the order-$k$ Voronoi diagram of a set of $n$ sites in the plane is in $O(k(n-k))$. The properties described in this paper allow us to give a new proof of this result.

2 The order-$k$ Voronoi faces

Given a set $S$ of $n$ sites in the $d$-dimensional Euclidean space $E$, the construction of the order-$k$ Voronoi diagram of $S$ consists in finding, for every point $x$ of $E$, the set $P$ of $k$ nearest sites of $x$. If the set $P$ exists, $x$ is the center of a sphere whose interior contains the $k$ sites of $P$ but no other site of $S$. Such a set does not exist if the $k^{th}$ and $(k+1)^{th}$ nearest sites of $x$ are at the same distance from $x$. In this case, $x$ is the center of a sphere that passes through a set $Q$ of sites and whose interior contains a set $P$ of sites such that $|P| < k < |P| + |Q|$. This leads to the following definitions:

A $k$-section of $S$ is a couple $(P, Q)$ of subsets of $S$ for which there exists an open ball $\omega$ with boundary $\delta(\omega)$ such that $P = \omega \cap S$, $Q = \delta(\omega) \cap S$, and such that $|P| = k$ if $Q$ is empty and $|P| < k < |P| + |Q|$ otherwise. The center of $\omega$ is called a center of the $k$-section $(P, Q)$ and the set $f_P(Q)$ of centers of $(P, Q)$ is called an order-$k$ Voronoi face.

Thus, by denoting $d(x, T)$ (resp. $d_{\text{max}}(x, T)$) the minimum (resp. maximum) distance from a point $x$ of $E$ to the sites of a subset $T$ of $S$, $f_P(Q)$ is the set of points of $E$ such that,

$$f_P(Q) = \{x \in E; d_{\text{max}}(x, P) < d(x, Q) = d_{\text{max}}(x, Q) < d(x, S \setminus (P \cup Q))\}$$

if $P$, $Q$ and $S \setminus (P \cup Q)$ are not empty. If $Q$ is empty, we get the classical definition of the order-$k$ Voronoi region of $P$:

$$f_P(\emptyset) = \{x \in E; d_{\text{max}}(x, P) < d(x, S \setminus P)\}.$$

Since every point $x$ of $E$ is the center of one and only one $k$-section of $S$, the set of order-$k$ Voronoi faces forms a partition of $E$ called the order-$k$ Voronoi diagram of $S$ (see Figure 1 for an illustration).

Note that we have not yet proven that the previously defined Voronoi faces of dimension lower than $d$ are the faces of the order-$k$ Voronoi regions. This proof will be given in Section 4.

For every subset $F$ of $E$, let $\text{dim}(F)$ be the dimension of the subspace spanned by $F$ and, for every subset $Q$ of cospherical sites of $S$, let $\text{bis}(Q) = \{x \in E; d(x, Q) = d_{\text{max}}(x, Q)\}$ be the bisector of $Q$.

**Theorem 1.** For every $k$-section $(P, Q)$ of $S$,

(i) if $Q = \emptyset$, then $f_P(Q)$ is an open connected convex $d$-dimensional region of $E$;

(ii) if $0 < \text{dim}(Q) < d$, then $f_P(Q)$ is an open connected convex subset of $\text{bis}(Q)$ and $\text{dim}(f_P(Q)) = d - \text{dim}(Q)$;

(iii) if $\text{dim}(Q) = d$, then $f_P(Q)$ is a point of $E$ (called an order-$k$ Voronoi vertex).

Proof. (i) $f_P(\emptyset) = \{x \in E; d_{\text{max}}(x, P) < d(x, S \setminus P)\}$ is the intersection of the open half spaces $\text{half}(p, s) = \{x \in E; d(x, p) < d(x, s)\}$ where $p$ is a site of $P$ and $s$ a site of $S \setminus P$. Thus, $f_P(\emptyset)$ is an open connected convex $d$-dimensional subset of $E$. 