Sparse Convolution Quadrature for Time Domain Boundary Integral Formulations of the Wave Equation by Cutoff and Panel–Clustering

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Summary. We consider the wave equation in a time domain boundary integral formulation. To obtain a stable time discretization, we employ the convolution quadrature method in time, developed by Lubich. In space, a Galerkin boundary element method is considered. The resulting Galerkin matrices are fully populated and the computational complexity is proportional to $N \log^2 NM$, where $M$ is the number of spatial unknowns and $N$ is the number of time steps.

We present two ways of reducing these costs. The first is an a priori cutoff strategy, which allows to replace a substantial part of the matrices by 0. The second is a panel clustering approximation, which further reduces the storage and computational cost by approximating subblocks by low rank matrices.

1 Introduction

This paper is concerned with the numerical solution of the wave equation in an unbounded domain. Problems governed by the wave equation arise in many physical applications such as electromagnetic wave propagation or the computation of transient acoustic waves. When such problems are formulated in unbounded domains, the approach of \textit{retarded potentials} allows a transformation of partial differential equations into space-time integral equations on the bounded surface of the scatterer.

Although this approach goes back to the early 1960s (cf. [11]) the development of fast numerical methods for integral equations in the field of hyperbolic problems is still in its infancies compared to the vast of fast methods for elliptic boundary integral equations (cf. [24] and references therein). Existing numerical discretisation methods include collocation methods with some stabilisation techniques (cf. [2, 3, 6, 7, 8, 22, 23]) and Laplace-Fourier
methods coupled with Galerkin boundary elements in space (cf. [1, 5, 9, 12]).
Numerical experiments can be found, e.g., in [13]. In [10], a fast version of the
marching-on-in-time (MOT) method is presented which is based on a suitable
plane wave expansion of the arising potential which reduces the storage and
computational costs.

In this paper, we consider the convolution quadrature method for the time
discretisation (cf. [18, 19, 20, 21]), and develop a panel-clustering method
to obtain a data-sparse approximation of the underlying boundary integral
equations. In [14], we have developed and analysed a simple cut-off strategy
which reduces the number of entries in the system matrix which have to be
computed while the rest is set to zero. The use of panel-clustering will further
reduce the storage and computational complexity.

In [25, 26, 27] Lubich’s convolution quadrature method is applied to
problems such as viscoelastic and poroelastic continua.

2 Formulation of the Problem

We consider a scattering problem in an exterior domain. For this, let \( \Omega \subset \mathbb{R}^3 \)
be an unbounded Lipschitz domain with boundary \( \Gamma \). Let \( \tilde{u} \) be the solution
to the wave equation

\[
\begin{align*}
\partial_t^2 \tilde{u} &= \Delta \tilde{u} + f \text{, in } \Omega \times (0, T), \\
\tilde{u}(\cdot, 0) &= u_0 \text{ in } \Omega, \\
\partial_t \tilde{u}(\cdot, 0) &= u_1 \text{ in } \Omega, \\
\tilde{u} &= 0 \text{ on } \Gamma \times (0, T),
\end{align*}
\]

for some time interval \((0, T)\) and given data \( f, u_0 \) and \( u_1 \).

To formulate the differential equation as a boundary integral equation, we
introduce an incident solution \( v \) and a diffracted solution \( u \) in the whole \( \mathbb{R}^3 \),
with \( \tilde{u}|_\Omega = (u + v)|_\Omega \), where \( v \) solves the open space
problem

\[
\begin{align*}
\partial_t^2 v &= \Delta v + f_p \text{ in } \mathbb{R}^3 \times (0, T), \\
v(\cdot, 0) &= u_{0p} \text{ in } \mathbb{R}^3, \\
\partial_t v(\cdot, 0) &= u_{1p} \text{ in } \mathbb{R}^3,
\end{align*}
\]

where \( f_p, u_{ip} \) are prolongations of \( f \) and \( u_i \) to the whole \( \mathbb{R}^3 \), respectively.
Given the solution to the above problem, \( v, u \) solves the homogeneous wave
equation

\[
\begin{align*}
\partial_t^2 u &= \Delta u \text{ in } \Omega \times (0, T), \\
u(\cdot, 0) &= \partial_t u(\cdot, 0) = 0 \text{ in } \Omega, \\
u &= g \text{ on } \Gamma \times (0, T),
\end{align*}
\]

where \( g = -v|_{\Gamma \times (0, T)} \).