Martingales and Stochastic Integrals

In this chapter we consider a class of continuous stochastic processes, called martingales, which play a central role in finance. We also define the gains realized from trading as a stochastic integral. Stochastic integration and martingales provide key tools for the analysis of the continuous time evolution of financial markets.

5.1 Martingales

One of the fundamental concepts in modern finance is the notion of a martingale. This is a stochastic process that, with its last observed value, provides the best forecast for its future values. Martingales exhibit the property of having no systematic trends in their dynamics. It is obvious that financial quantities, such as asset prices, are driven primarily by information. Forecasting a quantity, for example, the value of a derivative price when expressed in units of the market portfolio, is strongly dependent on the information that is available at the present time. This forces one to use a detailed notion for the information structure related to the evolution of the underlying stochastic processes.

Information Sets and Filtrations

On a given probability space \((\Omega, \mathcal{A}, P)\), as introduced in Sect. 1.1, let us consider a financial market model that is based on the observation of a continuous time stochastic vector process \(X = \{X_t \in \mathbb{R}^n, t \in [0, \infty)\}, n \in \mathbb{N}\), typically expressing asset price processes. We denote by \(\mathcal{A}_t\) the time \(t\) information set, which is the sigma-algebra of events that are known to the market participants at time \(t \in [0, \infty)\). Our interpretation of \(\mathcal{A}_t\) is that it represents the information obtained from the values of the vector process \(X\) up to time \(t\). More precisely, it is the sigma-algebra.

\[ \hat{A}_t = \sigma\{X_s : s \in [0, t]\} \]
generated from all observations of \( X \) in the market up to time \( t \). In a general financial market model the components of \( X \) could include diverse quantities, for instance, security prices, interest rates, indicators for certain political events, market activity, corporate data, employment figures, insurance claims, balance sheets of companies or trade balances.

Assuming that information is not lost, then the increasing family
\[ \hat{A} = \{\hat{A}_t, t \in [0, \infty)\} \]
of information sets \( \hat{A}_t \), which are sub-sigma-algebras of \( \hat{A}_\infty \) satisfy, for any sequence \( 0 \leq t_1 < t_2 < \ldots < \infty \) of observation times, the relation \( \hat{A}_{t_1} \subseteq \hat{A}_{t_2} \subseteq \ldots \subseteq \hat{A}_\infty = \cup_{t \in [0, \infty)} \hat{A}_t \).

Furthermore, to avoid technical subtleties, we introduce the information set \( \hat{A}_t \) as the augmented sigma-algebra of \( \hat{A}_t \) for each \( t \in [0, \infty) \). This means that it is augmented by every null set in \( \hat{A}_\infty \) such that it belongs to \( \hat{A}_0 \), and so to each \( \hat{A}_t \). We define \( \hat{A}_{t+} = \cap_{\varepsilon > 0} \hat{A}_{t+\varepsilon} \) to be the sigma-algebra of events immediately after \( t \in [0, \infty) \). We say that the family \( \hat{A} = \{\hat{A}_t, t \in [0, \infty)\} \) is right continuous if \( \hat{A}_t = \hat{A}_{t+} \) holds for every \( t \in [0, \infty) \). Such a right-continuous family \( \hat{A} = \{\hat{A}_t, t \in [0, \infty)\} \) of information sets we call a filtration. Thus, a filtration models the evolution of information as it becomes available over time. For simplicity, we define \( A \) as the smallest sigma-algebra that contains \( A_\infty = \bigcup_{t \in [0, \infty)} \hat{A}_t \).

The above technical assumptions allow convenient mathematical derivations and do not restrict our practical modeling potential. From now on, if not stated otherwise, we shall assume a filtered probability space \((\Omega, \mathcal{A}, \hat{A}, P)\) to be given, where the filtration \( \hat{A} \) characterizes the evolution of the corresponding information. The capturing of the evolution of this information is essential for the modeling of financial markets since it is information that drives most of its dynamics.

Any given stochastic process \( Y = \{Y_t, t \in [0, \infty)\} \) generates a filtration \( \mathcal{A}^Y = \{\mathcal{A}^Y_t, t \in [0, \infty)\} \). Here \( \mathcal{A}^Y_t = \sigma\{Y_s : s \in [0, t]\} \) is the information set, that is the sigma-algebra, generated by \( Y \) up to time \( t \). This information set can be interpreted as a complete record of all movements of the process \( Y \) up until time \( t \). \( \mathcal{A}^Y \) is also called the natural filtration for the process \( Y \). For a given model with a vector process \( X \) that describes the total evolution of the model and, thus, the corresponding increasing family of information sets, we write \( \hat{A} = \mathcal{A}^X \) and set \( \hat{A}_t = \mathcal{A}^X_t \), similarly as above.

If for a process \( Z = \{Z_t, t \in [0, \infty)\} \) and each time \( t \in [0, \infty) \) the random variable \( Z_t \) is \( \mathcal{A}^X_t \)-measurable, then \( Z \) is called adapted to \( \mathcal{A}^X = \{\mathcal{A}^X_t, t \in [0, \infty)\} \). In intuitive terms this means that the history of the process \( Z \) up until time \( t \) is covered by the information set \( \mathcal{A}^X_t \). As a consequence, for an \( \mathcal{A}^X \)-adapted process \( Z \) the value \( Z_t \) is known, given the information set \( \mathcal{A}^X_t \) up to and including time \( t \). We mention that the completeness of the information set \( \mathcal{A}^X_t \), which includes all null events, allows us to conclude that for two