The Itô Formula

The price of a security, for instance, a zero coupon bond which generates some future payoff at a maturity date, is often dependent on the value of an underlying process. In many applications, the effect of changes in the underlying process on this price needs to be quantified. In deterministic calculus this type of problem is handled by the chain rule. In stochastic calculus the corresponding generalization of the chain rule is given by the Itô formula. This stochastic chain rule contains terms reflecting the effect due to the stochastic processes involved having non-zero quadratic variation. In this chapter we introduce, apply and derive the Itô formula. It is widely regarded as the main tool in stochastic calculus and is therefore highly important in quantitative finance.

6.1 The Stochastic Chain Rule

The Classical Chain Rule

First consider an example, where the classical deterministic chain rule applies. Suppose we observe in the market the price of a savings account $B_t = \exp\{r t\}$, where $r$ denotes a constant continuously compounding interest rate. Then

$$dB_t = r B_t \, dt$$

(6.1.1)

for $t \in [0, \infty)$ with $B_0 = 1$. Also suppose that we are interested in a financial quantity $u(B_t)$, where $u: \mathbb{R} \rightarrow \mathbb{R}$ is some differentiable function. For instance, such a quantity could be the square of the value of the savings account, that is, $u(B_t) = (B_t)^2$. Furthermore, suppose that we need to express the evolution of this quantity in terms of properties of $u$ and $B$ with respect to time. In this case, by using the well-known chain rule of deterministic calculus, we can write the equations

$$u(B_t) = u(B_0) + \int_0^t u'(B_s) \, dB_s = u(B_0) + \int_0^t u'(B_s) \, r B_s \, ds$$

(6.1.2)
for \( t \in [0, \infty) \). Note from the first line in (6.1.2) that the value of the quantity \( u(B_t) \) can be interpreted as the gains from trade with integrand \( u'(B_t) \) and integrator \( B_t \) for \( t \in [0, \infty) \). This means for our simple deterministic example that

\[
(B_t)^2 = (B_0)^2 + 2 \int_0^t B_s dB_s
\]

(6.1.3)

for \( t \in [0, \infty) \).

A Stochastic Example

In Sect. 5.3 we considered the Itô integral

\[
I_{W,W}(t) = \int_0^t W_s dW_s,
\]

which is the double Wiener integral for a Wiener process \( W = \{W_t, t \in [0, \infty)\} \). This stochastic integral was interpreted as the gains from trade, where the number of shares held in the asset whose price was \( W \) was equal to its price. By rewriting equation (5.3.8) we obtain

\[
(W_t)^2 = 2 \int_0^t W_s dW_s + [W]_t = 2 \int_0^t W_s dW_s + \int_0^t ds
\]

(6.1.4)

for \( t \in [0, \infty) \). Using the Itô differentials \( dW_t \) and \( d(W_t)^2 \) the equation (6.1.4) can be expressed in the equivalent Itô differential form

\[
d(W_t)^2 = 2 W_t dW_t + dt
\]

(6.1.5)

for \( t \in [0, \infty) \) with initial value \((W_0)^2 = 0\). As previously explained, the equation (6.1.5) is nothing more than an abbreviated form of the stochastic integral equation (6.1.4). This integral equation involves an Itô integral, which is well defined, as discussed in the previous chapter. As a rule in stochastic calculus we shall see later that one can treat \((dW_t)^2\) as \(d[W]_t = dt\), see (5.4.5). Note however that \(d(W_t)^2\) is different to \((dW_t)^2\). Another rule will suggest setting \((dt)^2 = [\cdot]_t = 0\) and \(dW_t dt = d[W,t]_t = 0\).

Heuristic Derivation of the Itô Formula

One of the key features of the Itô integral with respect to the Wiener process is its martingale property, described in (5.4.3), which makes it an essential tool for pricing in finance. However, as previously indicated, this fundamental property does not come freely, namely the chain rule of classical calculus does not apply when using Itô integrals. Instead, the stochastic chain rule, the Itô formula, has to be applied. We now provide a heuristic derivation of the Itô formula. In Sect. 6.6 a proof of this formula will be presented.