14. Resolution 2

Throughout this section we will assume that \( \Phi_X : X \to S \) is weakly prepared.
We define a new condition on \( X \).

**Definition 14.1.** Suppose that \( r \geq 2 \). We will say that \( C_r(X) \) holds if:

1. If \( p \in X \) is a 1 point then \( \nu(p) \leq r \). If \( \nu(p) = r \) then \( \gamma(p) = r \).
2. If \( p \) is a 2 point then \( \nu(p) \leq r \). If \( \nu(p) = r \) then \( \gamma(p) = r \). If \( \nu(p) = r - 1 \) then one of the following three cases must hold:
   (a) \( \tau(p) > 0 \) or
   (b) \( \gamma(p) = r \) or
   (c) \( r \geq 3 \), \( \nu(p) = r - 1 \), \( \tau(p) = 0 \), \( p \notin \overline{S}_r(X) \), there exists a unique curve \( D \subset \overline{S}_{r-1}(X) \) containing a 1 point such that \( p \in D \), and permissible parameters \((x, y, z)\) at \( p \) such that \( x = z = 0 \) are local equations of \( D \),

\[
\begin{align*}
u &= (x^ay^b)^m \\
v &= P(x^ay^b) + x^ay^d F_p \\
F_p &= \tau x^{r-1} + \sum_{j=0}^{r-1} \overline{a}_j(y, z) y^j z^{r-1-j}
\end{align*}
\]

where \( \tau \) is a unit, \( \overline{a}_j \) are units (or 0). There exists \( i \) such that \( \overline{a}_i \neq 0 \),
\[
\left\{ \frac{d_j}{i} - \frac{e_i}{i} \right\} \leq \frac{e_j}{j}
\]
for all \( j \), and
\[
\left\{ \frac{d_i}{i} \right\} + \left\{ \frac{e_i}{i} \right\} < 1.
\]

3. If \( p \) is a 3 point then \( \nu(p) \leq r - 2 \).
4. \( \overline{S}_r(X) \) makes SNCs with \( \overline{B}_2(X) \).

**Remark 14.2.** If \( C_r(X) \) holds then there does not exist a 2 curve \( C \) on \( X \) such that \( C \) is \( r \) small or \( r-1 \) big.

In this section we will prove a condition stronger than \( C_r(X) \) (Theorem 14.7).

**Theorem 14.3.** Suppose that \( r \geq 2 \), \( A_r(X) \) holds, \( p \in X \) is a 2 point such that \( \nu(p) = r \) and \( 2 \leq \tau(p) < r \), then either

1. There exists a sequence of quadratic transforms \( \pi : Y \to X \) over \( p \) such that
   (a) \( A_r(Y) \) holds.
   (b) If \( q \in \pi^{-1}(p) \) is a 1 point then \( \nu(q) \leq r \). \( \nu(q) = r \) implies \( \gamma(q) = r \).
   (c) If \( q \in \pi^{-1}(p) \) is a 2 point then \( \nu(q) \leq r - 1 \).
   (d) If \( q \in \pi^{-1}(p) \) is a 3 point, then \( \nu(q) \leq r - 2 \).
   (e) If \( D \subset \pi^{-1}(p) \) is a 2 curve, then \( D \) is not \( r \) small or \( r-1 \) big.

or

2. There exists a curve \( C \subset \overline{S}_r(X) \) such that \( p \in C \) and \( C \) is \( r \) big at \( p \).

There exists an affine neighborhood \( U \) of \( p \) such that the blow-up of \( C \cap U \)
\[ \pi : Y \to U \] is a permissible monoidal transform such that
(a) \( A_r(Y) \) holds.

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(b) If \( q \in \pi^{-1}(p) \) is a 2 point, then \( \nu(q) \leq r - 1 \).
(c) If \( q \in \pi^{-1}(p) \) is the 3 point, then \( \nu(q) \leq r - 2 \).
(d) The 2 curve \( D = \pi^{-1}(p) \) is not \( r \) small or \( r \)-1 big.

In either case, if \( X \) satisfies the conclusions of Theorem 13.8, then \( Y \) satisfies the conclusions of Theorem 13.8.

**Proof.** \( p \) has permissible parameters \((x, y, z)\) such that
\[
\begin{align*}
\nu &= (x^n y^k)^m \\
v &= P(x^n y^k) + x^c y^d F_p \\
F_p &= \sum_i \tau_{i+j+k \geq r} a_{i j k} x^i y^j z^k
\end{align*}
\]

Suppose that there does not exist a curve \( C \subset T_r(X) \) such that \( C \) is \( r \) big at \( p \).

Let \( \pi : X_1 \rightarrow X \) be the blow-up of \( p \). We will first show that (a), (b) and (d) of 1. hold on \( X_1 \) and if \( q \in \pi^{-1}(p) \) is a 2 point with \( \nu(q) = r \) then \( \tau(q) \geq \tau(p) \). This follows from Theorem 7.1, Theorem 7.3 and Lemma 7.9. All exceptional 2 curves \( D \) of \( \pi \) contain a 3 point \( q \) such that \( \nu(q) \leq r - 2 \). (c) thus holds by Lemmas 8.1 and 7.7.

By Lemma 8.1 there are at most finitely many 2 points \( q \in \pi^{-1}(p) \) such that \( \nu(q) = r \). Suppose that there exists a 2 point \( q \in \pi^{-1}(p) \) and \( \nu(q) = r \). After a permissible change of parameters at \( p \), we have permissible parameters \((x_1, y_1, z_1)\) at \( q \) such that \( x = x_1, y = x_1 y_1, z = x_1 z_1 \). \( L_p = L_p(y, z) \) depends on both \( y \) and \( z \).

Suppose there also exists a 2 point \( q' \in \pi^{-1}(p) \) such that \( \nu(q') = r \) and \( q' \) has permissible parameters \((x', y', z')\) such that
\[
x = x' y', y = y', z = y'(z' + \alpha)
\]
for some \( \alpha \in k \). Then there exists a form \( L(x, z - \alpha y) \) such that
\[
L_p(y, z) = \begin{cases} 
L(x, z - \alpha y) + \tau x^n y^k & \text{if there exists } \alpha, \beta \in \mathbb{N} \text{ such that } \\
L(x, z - \alpha y) & \text{otherwise}
\end{cases}
\]

Thus
\[
L_p = \overline{d}(z - \alpha y)^r + \tau y^r
\]
for some \( \overline{d}, \tau \in k \) with \( \overline{d} \neq 0 \), a contradiction to the assumption that \( \tau(p) < r \).

Let
\[
\cdots \rightarrow Y_n \rightarrow Y_{n-1} \rightarrow \cdots \rightarrow Y_1 \rightarrow X
\]
be the sequence of quadratic transforms \( \pi_n : Y_n \rightarrow Y_{n-1} \) constructed by blowing up all 2 points \( q' \) on \( \pi_n \) which lie over \( p \) and have \( \nu(q') = r \).

Suppose that this sequence has infinite length. Then there exists \( q_n \in Y_n \) such that \( \pi_n(q_n) = q_{n-1} \) and \( \nu(q_n) = r \) for all \( n \). There exists a series \( \phi(x) = \sum \alpha_i x^i \) such that after replacing \( z \) with \( z - \phi(x) \), \( q_n \) has permissible parameters \((x_n, y_n, z_n)\) such that
\[
x = x_n, y = x_n^n y_n, z = x_n^n z_n
\]
and
\[
F_{q_n} = L_q(y_n, z_n) + x_n \Omega_n.
\]