14. Resolution 2

Throughout this section we will assume that \( \Phi_X : X \to S \) is weakly prepared. We define a new condition on \( X \)

**Definition 14.1.** Suppose that \( r \geq 2 \). We will say that \( C_r(X) \) holds if:

1. If \( p \in X \) is a 1 point then \( \nu(p) \leq r \). If \( \nu(p) = r \) then \( \gamma(p) = r \).
2. If \( p \) is a 2 point then \( \nu(p) \leq r - 2 \). If \( \nu(p) = r \) then \( \gamma(p) = r - 1 \) then one of the following three cases must hold:
   (a) \( \tau(p) > 0 \) or
   (b) \( \gamma(p) = r \) or
   (c) \( r \geq 3, \nu(p) = r - 1, \tau(p) = 0, p \notin \mathbb{S}_r(X), \) there exists a unique curve \( D \subset \mathbb{S}_{r-1}(X) \) containing a 1 point such that \( p \in D \), and permissible parameters \((x, y, z)\) at \( p \) such that \( x = z = 0 \) are local equations of \( D \),
      \[
      \begin{align*}
      u &= (x^a y^b)^m \\
      v &= P(x^a y^b) + x^j y^d F_p \\
      F_p &= \tau x^{r-1} + \sum_{j=1}^{r-1} a_j (y, z) y^{j} x^{r-1-j}
      \end{align*}
      \]
      where \( \tau \) is a unit, \( a_j \) are units (or 0). There exists \( i \) such that \( a_i \neq 0 \), \( e_i = i, 0 < d_i < i, \)
      \[
      \frac{d_i}{i} \leq \frac{d_j}{j} \frac{e_i}{i} \leq \frac{e_j}{j}
      \]
      for all \( j \), and
      \[
      \left\{ \frac{d_i}{i} \right\} + \left\{ \frac{e_i}{i} \right\} < 1.
      \]
3. If \( p \) is a 3 point then \( \nu(p) \leq r - 2 \).
4. \( \mathbb{S}_r(X) \) makes SNCs with \( \mathcal{B}_2(X) \).

**Remark 14.2.** If \( C_r(X) \) holds then there does not exist a 2 curve \( C \) on \( X \) such that \( C \) is \( r \) small or \( r-1 \) big.

In this section we will prove a condition stronger than \( C_r(X) \) (Theorem 14.7).

**Theorem 14.3.** Suppose that \( r \geq 2, A_r(X) \) holds, \( p \in X \) is a 2 point such that \( \nu(p) = r \) and \( 2 \leq \tau(p) < r \), then either

1. There exists a sequence of quadratic transforms \( \pi : Y \to X \) over \( p \) such that
   (a) \( A_r(Y) \) holds.
   (b) If \( q \in \pi^{-1}(p) \) is a 1 point then \( \nu(q) \leq r \). \( \nu(q) = r \) implies \( \gamma(q) = r \).
   (c) If \( q \in \pi^{-1}(p) \) is a 2 point then \( \nu(q) \leq r - 1 \).
   (d) If \( q \in \pi^{-1}(p) \) is a 3 point, then \( \nu(q) \leq r - 2 \).
   (e) If \( D \subset \pi^{-1}(p) \) is a 2 curve, then \( D \) is not \( r \) small or \( r-1 \) big.
   or
2. There exists a curve \( C \subset \mathbb{S}_r(X) \) such that \( p \in C \) and \( C \) is \( r \) big at \( p \).
   There exists an affine neighborhood \( U \) of \( p \) such that the blow-up of \( C \cap U \), \( \pi : Y \to U \) is a permissible monoidal transform such that
   (a) \( A_r(Y) \) holds.

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(b) If \( q \in \pi^{-1}(p) \) is a 2 point, then \( \nu(q) \leq r - 1 \).
(c) If \( q \in \pi^{-1}(p) \) is the 3 point, then \( \nu(q) \leq r - 2 \).
(d) The 2 curve \( D = \pi^{-1}(p) \) is not \( r \) small or \( r \)-1 big.

In either case, if \( X \) satisfies the conclusions of Theorem 13.8, then \( Y \) satisfies the conclusions of Theorem 13.8.

**Proof.** \( p \) has permissible parameters \((x, y, z)\) such that

\[
\begin{align*}
u &= (x^a y^b)^m, \\
v &= P(x^a y^b) + x^c y^d F_p, \\
F_p &= \sum_{i+j+k \geq r} a_{ijk} x^i y^j z^k
\end{align*}
\]

Suppose that there does not exist a curve \( C \subset \mathfrak{F}_r(X) \) such that \( C \) is \( r \) big at \( p \).

Let \( \pi : X_1 \to X \) be the blow-up of \( p \). We will first show that (a), (b) and (d) of 1. hold on \( X_1 \) and if \( q \in \pi^{-1}(p) \) is a 2 point with \( \nu(q) = r \) then \( \tau(q) \geq \tau(p) \).

This follows from Theorem 7.1, Theorem 7.3 and Lemma 7.9. All exceptional 2 curves \( D \) of \( \pi \) contain a 3 point \( q \) such that \( \nu(q) \leq r - 2 \). (c) thus holds by Lemmas 8.1 and 7.7.

By Lemma 8.1 there are at most finitely many 2 points \( q \in \pi^{-1}(p) \) such that \( \nu(q) = r \). Suppose that there exists a 2 point \( q \in \pi^{-1}(p) \) and \( \nu(q) = r \).

After a permissible change of parameters at \( p \), we have permissible parameters \((x_1, y_1, z_1)\) at \( q \) such that \( x = x_1, y = x_1 y_1, z = x_1 z_1 \), \( L_p = L_p(y, z) \) depends on both \( y \) and \( z \).

Suppose there also exists a 2 point \( q' \in \pi^{-1}(p) \) such that \( \nu(q') = r \) and \( q' \) has permissible parameters \((x', y', z')\) such that

\[
x = x', y = y', z = y'(z' + \alpha)
\]

for some \( \alpha \in k \). Then there exists a form \( L(x, z - \alpha y) \) such that

\[
L_p(y, z) = \begin{cases} 
L(x, z - \alpha y) + \overline{\tau} x^\overline{\pi} y^{\overline{\beta}} & \text{if there exists } \overline{\pi}, \overline{\beta} \in \mathbb{N} \text{ such that } \\
\overline{\pi} + \overline{\beta} = r, \alpha(d + \overline{\beta}) - b(c + \overline{\pi}) = 0 \\
L(x, z - \alpha y) & \text{otherwise}
\end{cases}
\]

Thus

\[
L_p = \overline{d}(z - \alpha y)^r + \overline{\tau} y^r
\]

for some \( \overline{d}, \overline{\tau} \in k \) with \( \overline{d} \neq 0 \), a contradiction to the assumption that \( \tau(p) < r \).

Let

\[
\cdots \to Y_n \to Y_{n-1} \to \cdots \to Y_1 \to X
\]

be the sequence of quadratic transforms \( \pi_n : Y_n \to Y_{n-1} \) constructed by blowing up all 2 points \( q' \) on \( Y_n \) which lie over \( p \) and have \( \nu(q') = r \).

Suppose that this sequence has infinite length. Then there exists \( q_n \in Y_n \) such that \( \pi_n(q_n) = q_{n-1} \) and \( \nu(q_n) = r \) for all \( n \). There exists a series \( \phi(x) = \sum \alpha_i x^i \) such that after replacing \( z \) with \( z \), \( q_n \) has permissible parameters \((x_n, y_n, z_n)\) such that

\[
x = x_n, y = x_n^a y_n, z = x_n^{-a}
\]

and

\[
F_{q_n} = L_q(y_n, z_n) + x_n^{\Omega_n}.
\]