15. Resolution 3

Throughout this section we will assume that $\Phi_X : X \to S$ is weakly prepared.

**Lemma 15.1.** Suppose that $C \subset X$ is a 2 curve. Suppose that $t$ is a natural number or $\infty$. Then the set

$$\{ q \in C \mid q \text{ is a 2 point and } \gamma(q) \geq t \}$$

is Zariski closed in $C - B_3(X)$.

**Proof.** Suppose that $p \in C$ is a 2 point. There exist permissible parameters $(x, y, z)$ at $p$ such that $(x, y, z)$ are uniformizing parameters in an \'{e}tale neighborhood $U$ of $p$ in $X$. At $p$,

$$u = (x^ay^b)^m$$
$$v = P(x^ay^b) + x^cy^dF(x, y, z).$$

Set

$$w = \frac{v - P(x^ay^b)}{x^cy^d}$$

with $\lambda > c + d$. $w \in \Gamma(U, \mathcal{O}_X)$. If $q \in C \cap U$, there are permissible parameters $(x, y, z_q = z - \alpha)$ at $q$ for some $\alpha \in k$. There exist $a_i(q) \in k$ such that

$$F_q = w - \sum_{i=0}^{\infty} a_i(q) \frac{(x^ay^b)^i}{x^cy^d}.$$ 

$$\{ q \in C \cap U \mid \nu(F_q(0, 0, z_q) \geq t) \} = \begin{cases} \{ q \in C \cap U \mid \frac{\partial w}{\partial x}(0, 0, \alpha) = 0, 0 \leq i < t \} & \text{if } ad - bc \neq 0 \\ \{ q \in C \cap U \mid \frac{\partial w}{\partial x}(0, 0, \alpha) = 0, 0 < i < t \} & \text{if } ad - bc = 0 \end{cases}$$

is Zariski closed.

Since $U$ is an \'{e}tale cover of an affine neighborhood $V$ of $p$,

$$\{ q \in C \mid q \text{ is a 2 point and } \gamma(q) \geq t \} \cap V$$

is Zariski closed in $V \cap C$. \hfill \Box

**Lemma 15.2.** Suppose that $C$ is a 2 curve and there exists $p \in C$ with permissible parameters $(x_p, y_p, z_p)$ at $p$ such that $x_p = y_p = 0$ are local equations of $C$ at $p$ and $\nu(F_p(0, 0, z_p)) < \infty$. If $q \in C$ then $\nu(F_q(0, 0, z_q)) < \infty$, where $(x_q, y_q, z_q)$ are permissible parameters at $q$ and $x_q = 0, y_q = 0$ are local equations for $C$ at $q$.

**Proof.** If $\nu(F_q(0, 0, z_q)) = \infty$, then $F_q \in \tilde{I}_{C, q}$ so that $F_p \in \tilde{I}_{C, p}$ for all $p \in C$ by Lemma 8.1. Thus $\nu(F_p(0, 0, z_p)) = \infty$ for all $p \in C$, a contradiction. \hfill \Box

**Theorem 15.3.** Suppose that $C_r(X)$ holds with $r \geq 2$ and the conclusions of Theorem 14.7 hold on $X$. Then there exists a sequence of permissible monoidal transforms $\pi : Y \to X$ centered at $r$ big curves $C$ in $\overline{S}_p$ such that $C_r(Y)$ and the conclusions of Theorem 14.7 hold on $Y$ and if $D$ is a curve in $\overline{S}_r(Y)$, then $D$ is not $r$ big.


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Proof. Suppose that the \( C \subset \overline{S}_r(X) \) is \( r \) big. \( C \) must contain a 1 point. Let \( \pi : Y \rightarrow X \) be the blow-up of \( C \).

By Lemma 8.8, \( C_r(Y) \) holds and the conclusions of Theorem 14.7 hold on \( Y \). There is at most one curve \( D \subset \overline{S}_r(Y) \cap \pi^{-1}(C) \). If this curve exists it must be a section over \( C \).

Let \( p \in C \) be a 1 point. As in (71) of the proof of Lemma 8.8, there exist permissible parameters \((x, y, z)\) at \( p \) such that \( \hat{I}_{C, p} = (x, z) \),

\[
\begin{align*}
  u &= x^a \\
  F_p &= \tau z^r + \sum_{i=2}^{r} a_i(x, y)z^{r-i}
\end{align*}
\] (173)

where \( \tau \) is a unit, \( x^i \mid a_i \) for \( 2 \leq i \leq r \).

As shown in the proof of Lemma 8.8, the only point \( q \in \pi^{-1}(p) \) which could be in \( \overline{S}_r(Y) \) is the 1 point with permissible parameters \( x = x_1, z = x_1z_1 \).

\[
\begin{align*}
  u &= x_1^a \\
  F_q &= \tau z_1^r + \sum_{i=2}^{r} a_i(x_1, y)z_1^{r-i}
\end{align*}
\] (174)

In this case, (174) has the form of (173) with

\[
\min\{ j i \mid a_i, x^j+1 \not\mid a_i \text{ for } 2 \leq i \leq r \}
\]

decreased by 1.

By induction on

\[
\min\{ j i \mid a_i, x^j+1 \not\mid a_i \text{ for } 2 \leq i \leq r \}
\]

we can construct a sequence of permissible blow-ups of \( r \) big curves in \( \overline{S}_r \) such that the conclusions of the Theorem hold. \( \square \)

Theorem 15.4. Suppose that \( C_r(X) \) holds with \( r \geq 2 \), the conclusions of Theorem 14.7 hold on \( X \) and if \( C \) is a curve in \( \overline{S}_r(X) \), then \( C \) is not \( r \) big. Suppose that \( p \in \overline{S}_r(X) \) is a 1 point, \( D \) is a general curve through \( p \). For a 1 point \( q \in D \), define

\[
\epsilon(D, q) = \nu(F_q(0, 0, z))
\]

where \((x, y, z)\) are permissible parameters at \( q \) so that \( \hat{I}_{D, q} = (x, y) \) and

\[
\begin{align*}
  u &= x^a \\
  v &= P(x) + x^cF_q.
\end{align*}
\]

Then there exists a sequence of blow-ups of points on the strict transform of \( D \), but not at \( p \), \( \lambda : Z \rightarrow X \) such that

1. \( C_r(Z) \) and the conclusions of Theorem 14.7 hold on \( Z \).
2. Let \( \tilde{D} \) be the strict transform of \( D \) on \( Z \). Then \( \epsilon(\tilde{D}, q) = 1 \) for all 1 points \( q \neq p \) on \( \tilde{D} \), \( \nu(q) = 0 \) if \( q \in \tilde{D} \) is a 2 point, and there are no 3 points on \( \tilde{D} \).
3. Suppose that \( \tilde{p} \) is a fundamental point of \( \lambda \).
   (a) If \( q \in \lambda^{-1}(\tilde{p}) \) is a 1 point then \( \nu(q) \leq r-1 \).